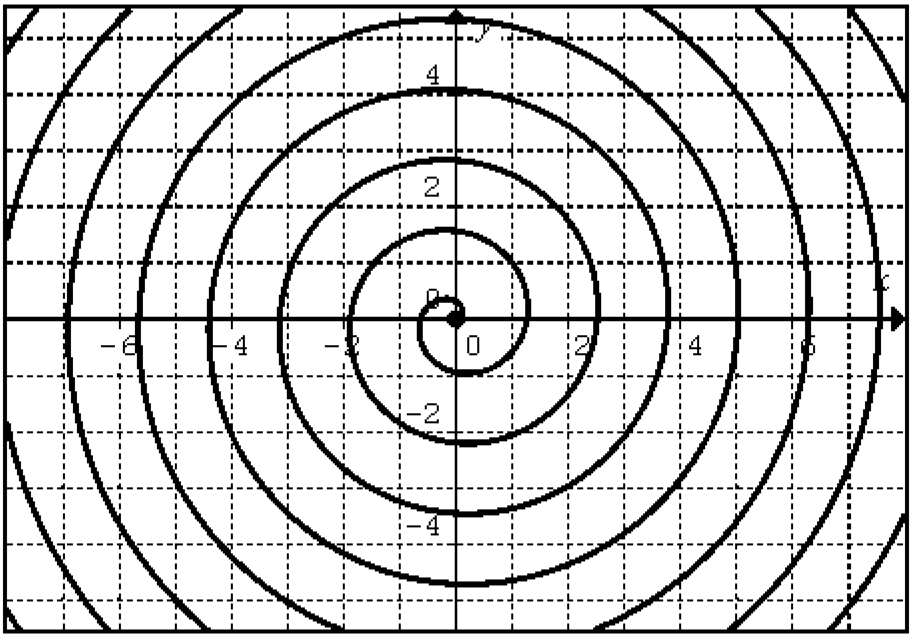
specialist mathematics



Stephen Zhang

VCE Specialist Mathematics Bound Reference – Stephen Zhang – 2015

Contents

Quick Reference

Graph Sketching

When sketching graphs, be sure to include:

- Labelled axes
- Intercepts labelled as coordinates (x,y) just to be sure
- Turning points and endpoints
- Label the function with $y = \cdots$

Prerequisites

Geometry

Pythagoras' Theorem

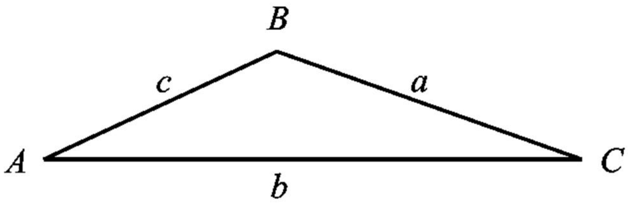
For a right angled triangle with hypotenuse c, stating that $c^2 = a^2 + b^2$

Converse Pythagoras' Theorem

If a triangle is such that $c^2 = a^2 + b^2$, then it is a right angled triangle with hypotenuse c.

Sine Rule

For a triangle ABC:

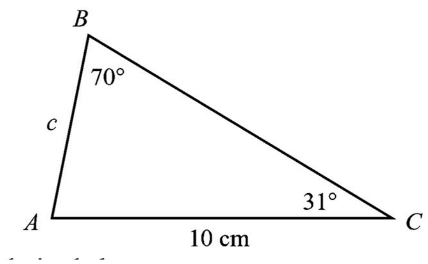


$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

The sine rule may be used when:

- One side and two angles are given (defining an unique triangle)
- Two sides and a non-included angle are given. (two possible triangles existing)

Example:



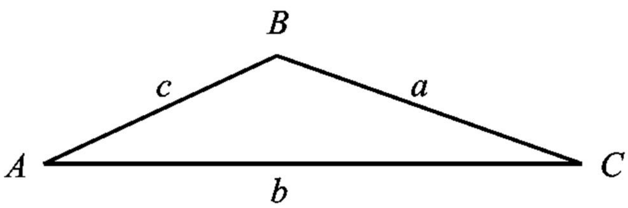
Find AB

$\frac{AB}{\sin(31^\circ)} = \frac{10}{\sin(70^\circ)}$

$$\therefore AB = \frac{10\sin(31^\circ)}{\sin(70^\circ)} = 5.481\ldots$$

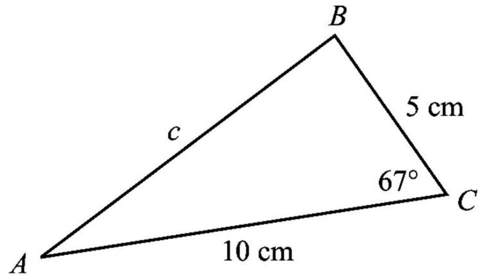
Cosine Rule

For a triangle ABC:



$$a^2 = c^2 + b^2 - 2bc\cos(A)$$

Example:

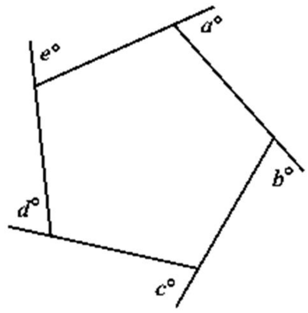


Find c.

$c^2 = 5^2 + 10^2 - 2(5)(10)\cos(67^\circ)$
$c^2 = 85.927\ldots$
$c = \sqrt{85.927\ldots} = 9.27\ldots$

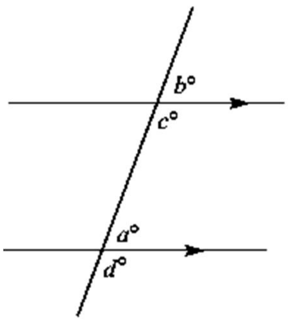
Other Theorems

- The sum of the interior angles of a triangle is 180
- The sum of exterior angles of a convex polygon is 360, i.e.



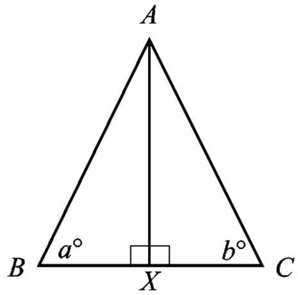
$$a + b + c + d + e = 360$$

- Corresponding angles of lines cut by a transversal iff they are parallel.



Here, a°/b° and c°/d° are corresponding angles

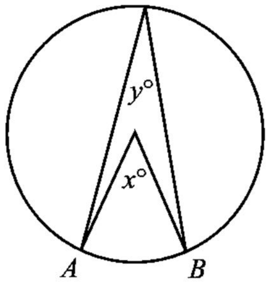
- Opposite angles of a parallelogram are equal, and opposite sides are equal and parallel.
- Base angles of an isosceles triangle are equal.
- For an isosceles triangle, the line joining the vertex to the midpoint of the base is perpendicular to the base, i.e. AX ⊥ BC
- For an isosceles triangle, the perpendicular bisector of the base passes through the vertex.



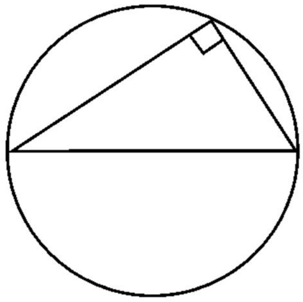
Circle Theorems

- The angle subtended by an arc at the centre of a circle is twice that subtended at the circumference.

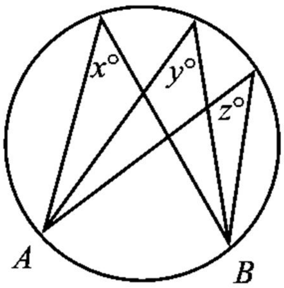
$x = 2y$



- The angle in a semicircle is a right angle.

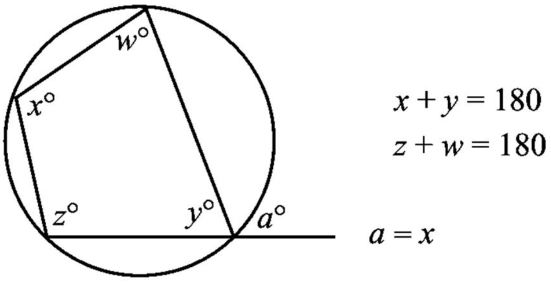


- Angles in the same segment of a circle are equal.

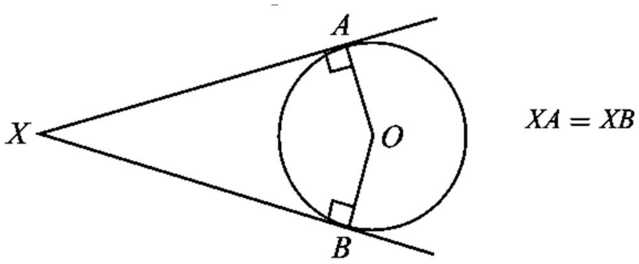


$x = y = z$

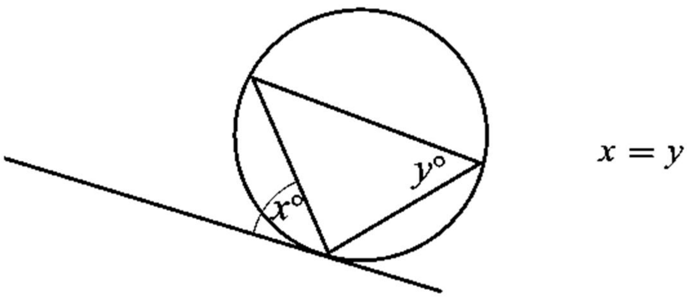
- The sum of opposite angles in a cyclic quadrilateral is 180.
- An exterior angle of a cyclic quadrilateral and the interior opposite angle are equal.



- A tangent to a circle is perpendicular to the radius at the point of tangency
- Two tangents to a circle from an exterior point are equal in length.



- The angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the alternate segment.



Series

Arithmetic Series

An arithmetic series is the sum of an arithmetic progression, that is, a progression of the form

$$a, a+d, a+2d, a+3d, \dots, a+(n-1)d$$

The nth term in the progression is given as

$$a_n = a+(n-1)d$$

The arithmetic series S_n, the sum of n terms of the progression is given by:

$$S_n = \sum_{i=1}^{n}[a+(i-1)d] = \frac{1}{2}(2a+(n-1)d) = \frac{n}{2}(a+l)$$

Where l is the last term in the series.

Geometric Series

A geometric series is the sum of a geometric progression, that is, a progression of the form

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}$$

The nth term in the progression is given as

$$a_n = ar^{n-1}$$

The geometric series S_n the sum of n terms of the progression is given by:

$$S_n = \sum_{i=1}^{n} ar^{i-1} = \begin{cases} \dfrac{a(1-r^n)}{1-r}, & |r|<1 \\ \dfrac{a(r^n-1)}{r-1}, & |r|>1 \end{cases}$$

Infinite series is the sum $S_\infty = \sum_{i=1}^{\infty} ar^{i-1}$, and is only defined if the series **converges**, that is, if $|r|<1$, $S_n \to S_\infty$, a finite value, as $n \to \infty$.

$$S_\infty = \frac{a}{1-r}, |r|<1$$

Conic Sections

Circles

General formula:

$$(x-h)^2 + (y-k)^2 = r^2$$

A circle centred at (h, k), with radius r.

Ellipses

General formula:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

An ellipse with horizontal radius a, vertical radius b, and centred at (h, k).

- The major axis of an ellipse is simply the larger diameter. It may be horizontal or vertical.
- The minor axis of an ellipse is likewise the smaller diameter.

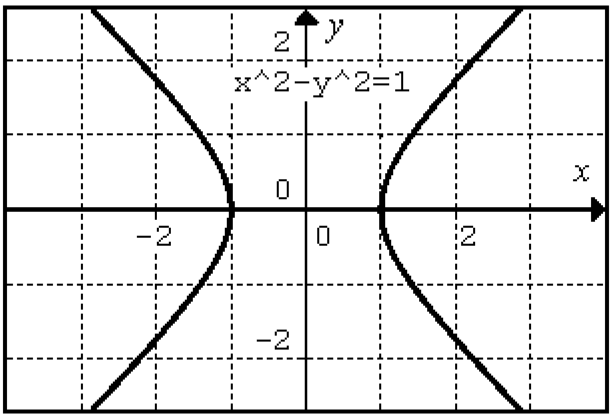
Definition of an ellipse**:** an ellipse with foci L_1, L_2 is the set of all points P such that $PL_1 + PL_2$ remain constant.

Hyperbolas

General formula:

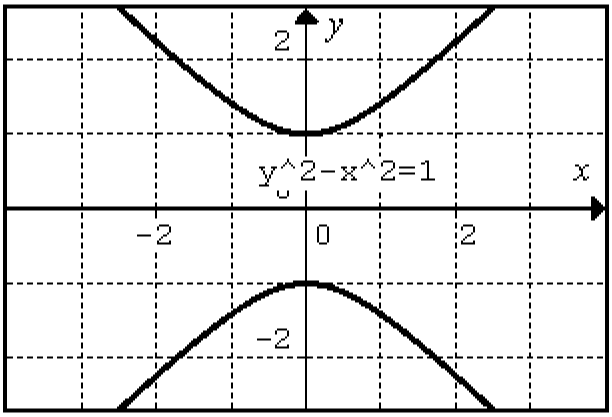
East-West

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



North-South

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$



Asymptotes for both cases:

$$y - k = \pm\frac{b}{a}(x-h)$$

Definition of a hyperbola: a hyperbola with foci L_1, L_2 is the set of all points P such that $PL_1 - PL_2$ remain constant.

Vectors

Vectors are mathematical quantities which have *direction* as well as *magnitude*. Scalars have *magnitude* only. For example, '5 km' is a scalar quantity, while '5km north' is a vector quantity.

A vector quantity is denoted by a tilde, i.e. $\underset{\sim}{r}$ or with an arrow, i.e. $\vec{r}$.

Free vectors have no starting point, and specify a displacement in a direction relative to the tail.

Position vectors have the origin O as their starting point.

Vector Properties

- **Equality** $-\ \vec{a} = \vec{b} \Leftrightarrow \vec{a}, \vec{b}$ have the same magnitude and direction.
- Commutative law for vector addition - $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative law for vector addition - $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- **Zero vector -** $\vec{a} + \vec{0} = \vec{a}$
- Additive inverse - $\vec{a} + (-\vec{a}) = \vec{0}$
- Associative law for scalar multiplication - $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$
- Two vectors are parallel iff $\vec{a} = k\vec{b}$, k ∈ ℝ

Special vectors

- The unit vector has magnitude 1: $\hat{\vec{r}} = \frac{\vec{r}}{|\vec{r}|}$
- The zero vector has magnitude zero and undefined direction, and is denoted $\vec{0}$

Example: Find a vector with magnitude 6, opposite to $2\vec{i} - 2\vec{j} + \vec{k}$

$$\underset{\sim}{r} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k} \qquad \therefore \hat{\underset{\sim}{r}} = \frac{2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{4+4+1}} = \frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$$

$$\therefore -6\underset{\sim}{r} = -\frac{6}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) = -4\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$$

$\vec{a} = 4\vec{i} + m\vec{j} - 18\vec{k}$, and $\vec{b} = -2\vec{i} + \vec{j} + n\vec{k}$ are parallel vectors. The values of m and n are:

$$\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 18\underset{\sim}{k}, \quad \underset{\sim}{b} = -2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k}$$

$\because \underset{\sim}{a}, \underset{\sim}{b}$ parallel $\quad \therefore \underset{\sim}{a} = k\underset{\sim}{b}, \quad k \in R$

$$\therefore 4\underset{\sim}{i} + m\underset{\sim}{j} - 18\underset{\sim}{k} = k(-2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k}) \qquad \therefore \begin{cases} 4 = -2k \\ m = k \\ -18 = nk \end{cases}$$

$$\therefore k = -2, \quad m = -2, \quad -18 = n(-2) \Rightarrow n = 9.$$

$$\therefore m = -2, \quad n = 9$$

Ratio Theorem

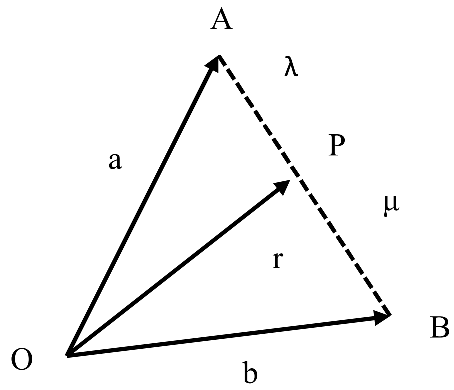
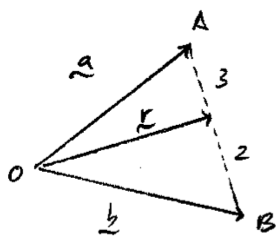


Figure 1 - Ratio theorem

Let $\vec{a}, \vec{b}$ be two nonparallel vectors. If P a point on AB, dividing AB in a ratio λ:μ, then

$$O\vec{P} = \frac{\mu\vec{a} + \lambda\vec{b}}{\lambda + \mu}$$

Example:



$\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$ $\underset{\sim}{b} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

Find $\underset{\sim}{r}$.

By ratio theorem:

$$\underset{\sim}{r} = \frac{2a + 3b}{3+2} = \frac{2(3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}) + 3(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})}{5}$$

$$= \frac{-3\underset{\sim}{i} + 10\underset{\sim}{j} + 24\underset{\sim}{k}}{5} = -\frac{3}{5}\underset{\sim}{i} + 2\underset{\sim}{j} + \frac{24}{5}\underset{\sim}{k}$$

<u>**Dot Product**</u>

The **dot product** (or *scalar product*) of two vectors is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$

Where θ is the angle between the two vectors, $0 \le \theta \le \pi$.

This is especially useful to find the angle between two vectors, in the form

$$\boxed{\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}}$$

Important special case – extremely useful to show that two vectors are perpendicular.

Two nonzero vectors are perpendicular if $\vec{a} \cdot \vec{b} = 0$. This is a handy property in vector proofs even when the vectors are geometrically defined.

Properties of the dot product

- The dot product is a *scalar* quantity, i.e. $\vec{a} \cdot \vec{b} \in \mathbb{R}$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (k\vec{b}) = (k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$
- $\vec{a} \cdot \vec{0} = 0$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} = 0 \bigvee \vec{b} = 0 \bigvee \vec{a} \perp \vec{b}$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

Cartesian Form

Vectors in Cartesian form can be expressed in terms of the unit vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ in the directions of the x, y, and z axes respectively.

General form - $\vec{r} = a\vec{\imath} + b\vec{\jmath}$ (in 2 dimensions) $\vec{r} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ (in 3 dimensions)

Dot product for Cartesian form

If $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

This also holds for vectors of any dimensions, in general:

If $\vec{a} = [a_1, a_2, \dots, a_n], \vec{b} = [b_1, b_2, \dots b_n]$, then $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$

Angles with the axes

- Angle with x-axis: $\cos(\theta) = \frac{\vec{a} \cdot \vec{\imath}}{|\vec{a}|}$
- Angle with y-axis: $\cos(\theta) = \frac{\vec{a} \cdot \vec{\jmath}}{|\vec{a}|}$
- Angle with z-axis: $\cos(\theta) = \frac{\vec{a} \cdot \vec{k}}{|\vec{a}|}$

Example: The angle between $\vec{x} = 3\vec{\imath} + 2\vec{\jmath} - 2\vec{k}$ and $\vec{y} = -\vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ is:

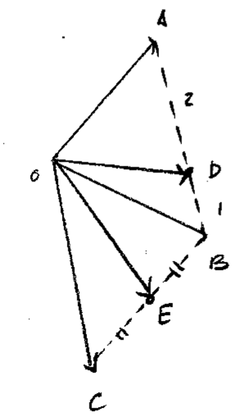
$\underset{\sim}{x} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}, \quad \underset{\sim}{y} = -\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$.

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{x} \cdot \underset{\sim}{y}}{|\underset{\sim}{x}||\underset{\sim}{y}|}\right) = \cos^{-1}\left(\frac{3(-1)+2(3)-2(-2)}{\sqrt{9+4+4}\sqrt{1+9+4}}\right) = \cos^{-1}\left(\frac{-3+6+4}{\sqrt{17}\sqrt{14}}\right) = \cos^{-1}\left(\frac{7}{\sqrt{17 \cdot 14}}\right)$$

$$\approx 51.498^\circ$$

Points A to E are in $\mathbb{R}^3$. $\overrightarrow{OA} = 2\vec{i} + 4\vec{j} + \vec{k}$, $\overrightarrow{OB} = -\vec{i} + \vec{j} + 4\vec{k}$, $\overrightarrow{OC} = 3\vec{i} + 3\vec{j} - 2\vec{k}$. D divides $\overrightarrow{AB}$ into a ratio of 2:1 and E is the midpoint of BC. Find $\overrightarrow{DE}$.

$\overrightarrow{OA} = 2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$, $\overrightarrow{OB} = -\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$, $\overrightarrow{OC} = 3\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$



$$\overrightarrow{OD} = \frac{2\overrightarrow{OB} + \overrightarrow{OA}}{3} = \frac{2(-\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) + 2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}}{3}$$

$$= \frac{\cancel{-2\underset{\sim}{i}} + 2\underset{\sim}{j} + 8\underset{\sim}{k} + \cancel{2\underset{\sim}{i}} + 4\underset{\sim}{j} + \underset{\sim}{k}}{3} = \frac{6\underset{\sim}{j} + 9\underset{\sim}{k}}{3} = 2\underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\overrightarrow{OE} = \frac{\overrightarrow{OB} + \overrightarrow{OC}}{2} = \frac{2\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}}{2} = \underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$$

$$\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = (\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) - (2\underset{\sim}{j} + 3\underset{\sim}{k}) = \underset{\sim}{i} - 2\underset{\sim}{k}$$

Linear Dependence and Independence

A property of a set of vectors:

- A set of vectors is *linearly dependent* if one of its members can be expressed as a combination of *scalar multiples* (not all zero) of the other members, i.e. the vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent if there exist $p, q, r \in \mathbb{R}$, not all zero, such that
$$p\vec{a} + q\vec{b} + r\vec{c} = \vec{0}$$
- A set of vectors is *linearly independent* if it is not linearly dependent.

Important facts include:

- A set of two vectors is **dependent** only if the vectors are parallel
- Any set of vectors containing zero vector is dependent
- In $\mathbb{R}^n$, any set containing more than n vectors is dependent.

If $\vec{a}, \vec{b}, \vec{c}$ are independent, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

I.e. $\vec{a}, \vec{b}, \vec{c}$ are coplanar and in $\mathbb{R}^3$, so they are linearly dependant (*this is because the matrix does not have full rank ⇒ there is a zero column, so determinant=0).* (See next section)

Knowing that vectors are independent can be useful in vector proofs.

Handy hint: for any set of vectors, the set is independent if the matrix formed by writing those vectors as columns has full rank

Coplanar Vector Test

A set of three vectors in $\mathbb{R}^3$ is coplanar if the *scalar triple product* is zero, that is:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Vector Resolutes

Vector resolution (or *vector projection*) is the act of splitting a vector into its **components** with respect to another vector.

Scalar Resolute Parallel

Let $\vec{a}, \vec{b}$ be two vectors. Then the scalar resolute parallel of $\vec{a}$ on $\vec{b}$ is given by:

$$(\vec{a}.\hat{b})$$

Vector Resolute Parallel

$$(\vec{a}.\hat{b})\hat{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{b}.\vec{b}}\vec{b}$$

Scalar Resolute Perpendicular

$$|\vec{a} - (\vec{a}.\hat{b})\hat{b}| = \left|\vec{a} - \frac{\vec{a}.\vec{b}}{\vec{b}.\vec{b}}\vec{b}\right|$$

Vector Resolute Perpendicular

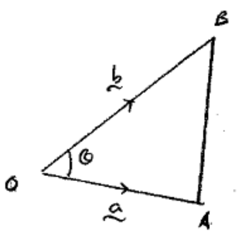
$$\vec{a} - (\vec{a}.\hat{b})\hat{b} = \vec{a} - \frac{\vec{a} \cdot \vec{b}}{\vec{b}.\vec{b}}\vec{b}$$

Vector Proofs

Vector properties can be very useful in proving some geometrical theorems, such as the cosine rule, the Pythagorean Theorem, and circle theorems. For these questions, the setup is essential, and must be as general as possible, assuming nothing.

Examples:

Prove the cosine rule.



Let OAB be any triangle, and denote $\vec{OA}$, $\vec{OB}$ as $\underset{\sim}{a}$, $\underset{\sim}{b}$ respectively.

$\therefore \vec{BA} = \underset{\sim}{a} - \underset{\sim}{b}$.

$\therefore \vec{BA}\cdot\vec{BA} = (\underset{\sim}{a} - \underset{\sim}{b})\cdot(\underset{\sim}{a} - \underset{\sim}{b})$

$= \underset{\sim}{a}\cdot\underset{\sim}{a} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b}$

$\therefore |\vec{BA}|^2 = |\underset{\sim}{a}|^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$

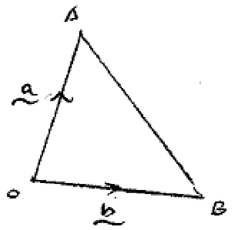
$= |\underset{\sim}{a}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta + |\underset{\sim}{b}|^2$

Let $|\vec{OB}| = b$, $|\vec{OA}| = a$, $|\vec{BA}| = c$.

$\therefore c^2 = a^2 + b^2 - 2ab\cos\theta$, the cosine rule

Prove the converse Pythagorean Theorem.

Suppose OAB is such that $AB^2 = OA^2 + OB^2$. Show that $OA \perp OB$.



Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$. $\therefore \overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$.

$\therefore \overrightarrow{AB} \cdot \overrightarrow{AB} = (\underset{\sim}{b} - \underset{\sim}{a})^2 = \underset{\sim}{b} . \underset{\sim}{b} - 2\underset{\sim}{a} . \underset{\sim}{b} + \underset{\sim}{a} . \underset{\sim}{a}$

$\therefore |\overrightarrow{AB}|^2 = |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} . \underset{\sim}{b} + |\underset{\sim}{a}|^2$

$\therefore AB^2 = OB^2 - 2\underset{\sim}{a} . \underset{\sim}{b} + OA^2$

But $AB^2 = OA^2 + OB^2$

$\therefore \cancel{OA^2} + \cancel{OB^2} = \cancel{OB^2} - 2\underset{\sim}{a} . \underset{\sim}{b} + \cancel{OA^2}$

$\therefore 0 = -2\underset{\sim}{a} . \underset{\sim}{b} = -2|\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\Rightarrow \cos\theta = 0 \quad (\because |\underset{\sim}{a}|, |\underset{\sim}{b}| \neq 0)$

$\Rightarrow \theta = \frac{\pi}{2}$

Cross Product (Not in study design)

The cross product is another type of vector product. The cross product of two vectors is a **vector**. The cross product of two vectors $\vec{a},\vec{b}$ is denoted $\vec{a}\times\vec{b}$.

Geometric Definition - $\vec{a}\times\vec{b}$ is the vector that is perpendicular to both $\vec{a}$ and $\vec{b}$.

Algebraic Definition

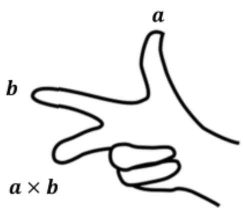
If $\vec{a}=(a_1,a_2,a_3),\vec{b}=(b_1,b_2,b_3)$:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\vec{\imath}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\vec{\jmath}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\vec{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

Properties

- $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$
- $\vec{a}\times\vec{a}=\vec{0}$
- $\vec{a}\times\vec{b}=\ -\vec{b}\times\vec{a}$
- $\vec{a}.\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$, the **scalar triple product**
- $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
- $\left(\vec{a}+\vec{b}\right)\times\left(\vec{c}+\vec{d}\right)=\vec{a}\times\vec{c}+\vec{a}\times\vec{d}+\vec{b}\times\vec{c}+\vec{b}\times\vec{d}$
- $\left|\vec{a}\times\vec{b}\right|=|\vec{a}||\vec{b}|\sin(\theta)$

Direction of $\vec{a}\times\vec{b}$



The direction of $\vec{a}\times\vec{b}$ is given by the **right hand rule**.

Geometric Interpretation

The area of a triangle defined by $\vec{a},\vec{b}$ is given as $A=\frac{1}{2}|\vec{a}\times\vec{b}|$

The area of a parallelogram defined by $\vec{a},\vec{b}$ is given as $A=\left|\vec{a}\times\vec{b}\right|$

The volume of the parallelepiped defined by $\vec{a},\vec{b},\vec{c}$ is given as $V=\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|$

Circular Functions

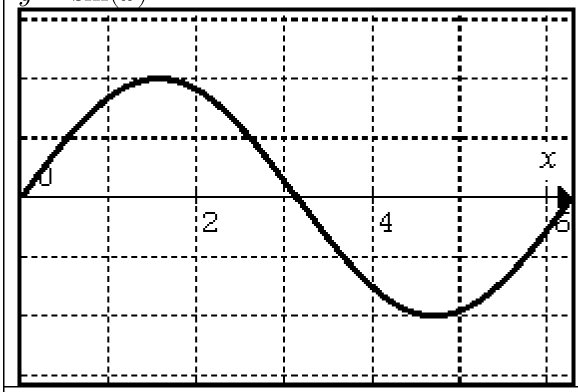
Trigonometric Functions

Function	*Definition*	*Period*	*Principal Domain*
$\sin(x)$		2π	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos(x)$		2π	$[0, \pi]$
$\tan(x)$	$\tan(x) = \frac{\sin(x)}{\cos(x)}$	π	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

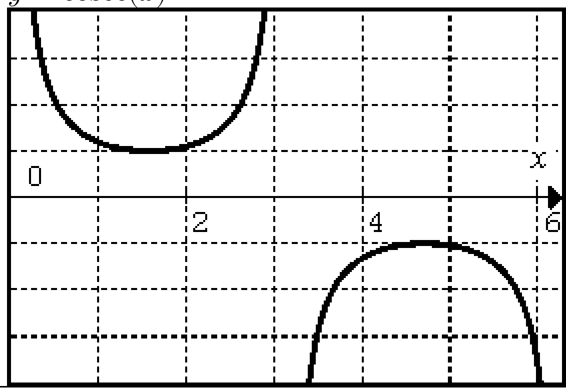
Reciprocal Trigonometric Functions

Function	*Definition*	*Period*
$\operatorname{cosec}(x)$	$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$	2π
$\sec(x)$	$\sec(x) = \frac{1}{\cos(x)}$	2π
$\cot(x)$	$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(\mathrm{x})}{\sin(\mathrm{x})}$	π

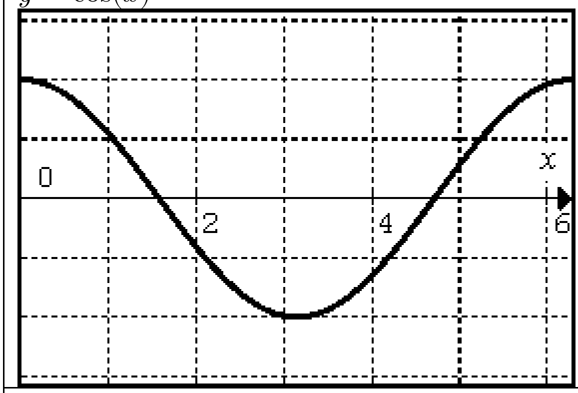
$y = \sin(x)$



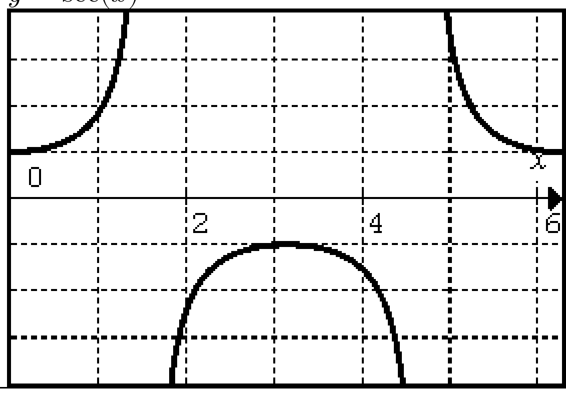
$y = \operatorname{cosec}(x)$



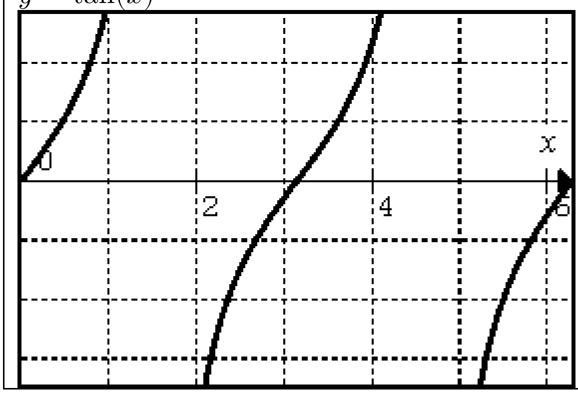
$y = \cos(x)$



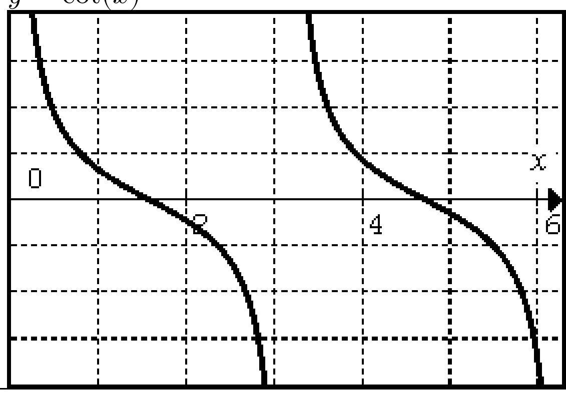
$y = \sec(x)$



$y = \tan(x)$



$y = \cot(x)$



Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Symmetrical identities

$$\sin(\pi - \theta) = \sin(\theta)$$

$$\cos(\pi - \theta) = -\cos(\theta)$$

$$\tan(\pi - \theta) = -\tan(\theta)$$

$$\sin(2\pi - \theta) = -\sin(\theta)$$

$$\cos(2\pi - \theta) = \cos(\theta)$$

$$\tan(2\pi - \theta) = -\tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$$

$$\sin(\pi + \theta) = -\sin(\theta)$$

$$\cos(\pi + \theta) = -\cos(\theta)$$

$$\tan(\pi + \theta) = \tan(\theta)$$

$$\sin(2\pi + \theta) = \sin(\theta)$$

$$\cos(2\pi + \theta) = \cos(\theta)$$

$$\tan(\pi + \theta) = \tan(\theta)$$

Compound Angle Formulae

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$\tan(x+y) = \frac{\tan(\mathrm{x}) + \tan(\mathrm{y})}{1 - \tan(\mathrm{x})\tan(\mathrm{y})}$$

$$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

Double Angle Formulae

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2 x}$$

Half Angle Formulae

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{\sin(x)}$$

Inverse Trigonometric Function Identity

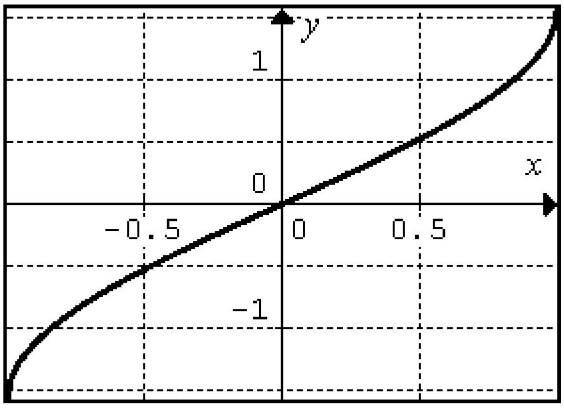
$$\arcsin(x) + \arccos(x) = \frac{\pi}{2}$$

Inverse Trigonometric Functions

These functions are the inverses of the restricted trigonometric functions, i.e. the trigonometric functions whose domains have been restricted to their principal domains.

Arcsine

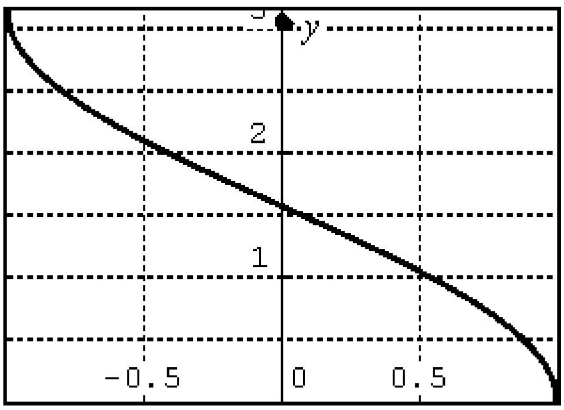
$$\arcsin(x) : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \sin(y) = x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



Points on graph: $(0,0), (-1, -\frac{\pi}{2}), (1, \frac{\pi}{2})$

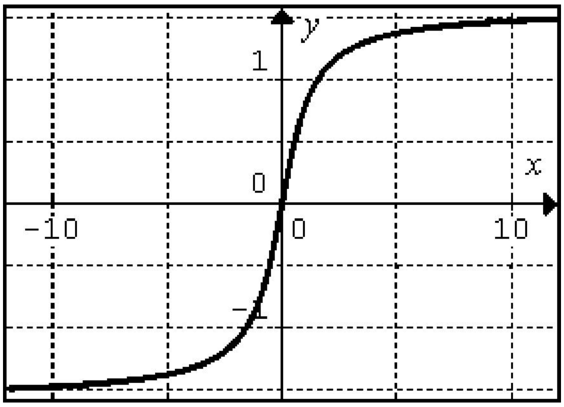
Arccosine

$$\arccos(x) : [-1,1] \to [0, \pi], \cos(y) = x, y \in [0, \pi]$$



Arctangent

$$\arctan(x) : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tan(y) = x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



Sketching Trigonometric Functions

##insert text##

Examples

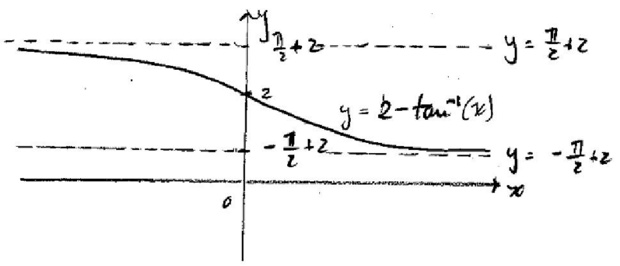
Sketch $y = 2 - \tan^{-1}(x)$, stating domain and range.

$y = 2 - \tan^{-1}(x)$,

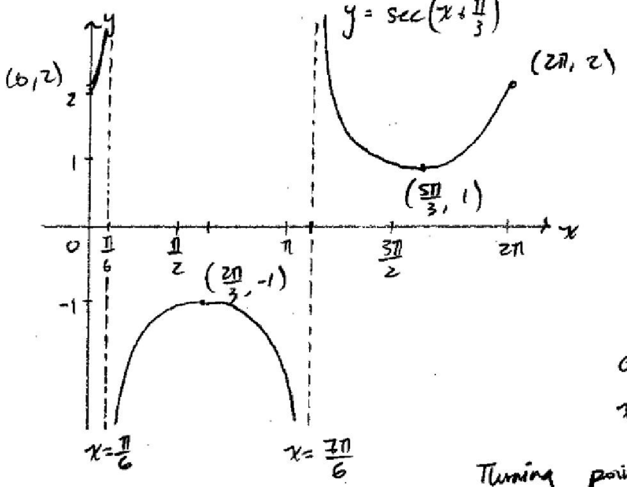
domain: $x \in \mathbb{R}$

range: $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow -\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow 2 - \tan^{-1}(x) \in \left(-\frac{\pi}{2} + 2, \frac{\pi}{2} + 2\right)$



Sketch $y = \sec\left(x + \frac{\pi}{3}\right)$, $0 \le x \le 2\pi$, stating equations of asymptotes, and coordinates of turning points.



asymptotes:
$x = \frac{\pi}{6}, \frac{7\pi}{6}$,

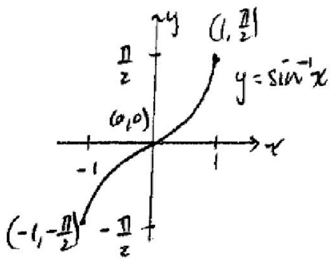
y-int:
$y|_{x=0} = \sec\left(\frac{\pi}{3}\right) = 2$.

Turning points:
$\sec\left(x + \frac{\pi}{3}\right) = 1$
$\therefore x + \frac{\pi}{3} = 0, 2\pi$
$\therefore x = -\frac{\pi}{3}, 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$
$\sec\left(x + \frac{\pi}{3}\right) = -1 \quad \therefore x + \frac{\pi}{3} = \pi$.

E8. Sketch $y = \sin^{-1}(2x+1)$, stating domain & range.

domain: $2x+1 \in [-1, 1] \Rightarrow 2x \in [-2, 0] \Rightarrow x \in$

range: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

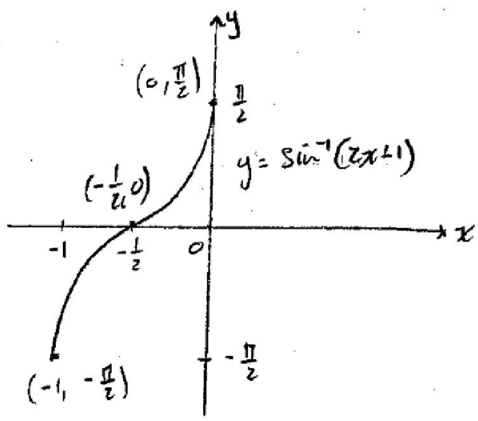


$y = \sin^{-1}(2x+1) = \sin^{-1}\left(2\left(x + \frac{1}{2}\right)\right)$

$\therefore \left(1, \frac{\pi}{2}\right) \rightarrow \left(\frac{1}{2}, \frac{\pi}{2}\right) \rightarrow \left(0, \frac{\pi}{2}\right)$

$\therefore (0, 0) \rightarrow (0, 0) \rightarrow \left(-\frac{1}{2}, 0\right)$

$\left(-1, -\frac{\pi}{2}\right) \rightarrow \left(-\frac{1}{2}, -\frac{\pi}{2}\right) \rightarrow \left(-1, -\frac{\pi}{2}\right)$



Solution of Trigonometric Equations

1. Rearrange for a simple equation such as $\sin(mx) = c$
2. Use inverse trig functions to find the values of x in the principal domain of the trig function (usually 2 for sin/cos, 1 for tan)
3. Add periods of the original equation to find all solutions

Example: Solve $\sin(\frac{3x}{7} + \frac{\pi}{9}) = \frac{1}{2}$ for $x \in [0, 8\pi]$

$$\sin\left(\frac{3x}{7} + \frac{\pi}{9}\right) = \frac{1}{2}$$
$$\therefore \frac{3x}{7} + \frac{\pi}{9} = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \frac{3x}{7} = \frac{\pi}{18}, \frac{13\pi}{18}$$
$$x = \frac{7\pi}{54}, \frac{91\pi}{54}$$

Period: $\frac{2\pi}{\frac{3}{7}} = \frac{14\pi}{3} = \frac{252\pi}{18}$

$$\therefore x = \frac{7\pi}{54}, \frac{91\pi}{54}, \frac{259\pi}{54}, \frac{343\pi}{54}$$

Examples:

(a) Show that $\frac{2\tan x}{1+\tan^2 x} = \sin(2x)$

(b) Hence find $\tan\frac{\pi}{12}$, in the form $m+n\sqrt{3}$, $m, n \in \mathbb{Z}$.

$$\frac{2\tan x}{1+\tan^2 x} = \frac{2\tan x}{\sec^2 x} = \frac{2\sin x}{\cancel{\cos x}} \times \cancel{\cos^2 x}^{\cos x} = 2\sin x\cos x = \sin(2x)$$

$$\frac{2\tan\frac{\pi}{12}}{1+\tan^2(\frac{\pi}{12})} = \sin\left(2\cdot\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

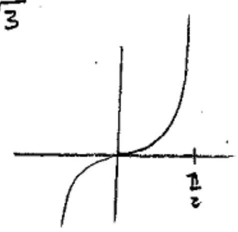
$$2\cdot 2\tan\frac{\pi}{12} = 1+\tan^2\frac{\pi}{12} \Rightarrow 4\tan\frac{\pi}{12} = 1+\tan^2\frac{\pi}{12}$$

$\therefore \tan^2\frac{\pi}{12} - 4\tan\frac{\pi}{12} + 1 = 0$. Let $t = \tan^2\frac{\pi}{12}$

$$\therefore t^2 - 4t + 1 = 0, \quad t = \frac{4 \pm \sqrt{16-4(1)}}{2} = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm\sqrt{12}}{2}$$
$$= \frac{4\pm 2\sqrt{3}}{2} = 2\pm\sqrt{3}$$

$\because \frac{\pi}{12} \in [0, \frac{\pi}{2}]$ $\therefore \tan\frac{\pi}{12} > 0$.

$\tan\frac{\pi}{12} < \tan\frac{\pi}{4} = 1$ $\therefore 2+\sqrt{3}$ incorrect.



$$\therefore \tan\frac{\pi}{12} = 2-\sqrt{3}$$

Solve $\sin(4x) = \cos(2x)$, $0 \le x \le \pi$.

$\sin(4x) = 2\sin(2x)\cos(2x) = \cos(2x)$

$\therefore 2\sin(2x)\cos(2x) - \cos(2x) = 0$

$\therefore \cos(2x)[2\sin(2x) - 1] = 0$.

$\therefore \cos(2x) = 0$ or $\sin(2x) = \frac{1}{2}$.

$\therefore 2x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad 2x = \frac{\pi}{6}, \frac{5\pi}{6}$.

$x = \frac{\pi}{4}, \frac{3\pi}{4}$ $\quad x = \frac{\pi}{12}, \frac{5\pi}{12}$

$\therefore x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$

Solution of Equations of the form $a\cos(x) + b\sin(x) = c$

Example: Solve $2\sin(\mathrm{x}) + 3\cos(\mathrm{x}) = 1$

$$2\sin(x) + 3\cos(x) = 1$$
$$\sqrt{2^2+3^2}\left(\frac{2}{\sqrt{2^2+3^2}}\sin(x) + \frac{3}{\sqrt{2^2+3^2}}\cos(x)\right) = 1$$
$$\sqrt{13}\left(\frac{2}{\sqrt{13}}\sin(x) + \frac{3}{\sqrt{13}}\cos(x)\right) = 1$$
$$\text{Suppose}\sin(\phi) = \frac{2}{\sqrt{13}}, 0 \le \phi \le \frac{\pi}{2}$$
$$\cos(\phi) = \pm\sqrt{1-\left(\frac{2}{\sqrt{13}}\right)^2} = \pm\sqrt{1-\frac{4}{13}} = \pm\sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}} \;\left(\because \phi \in \left[0, \frac{\pi}{2}\right]\right)$$
$$\sqrt{13}\left(\frac{2}{\sqrt{13}}\sin(x) + \frac{3}{\sqrt{13}}\cos(x)\right) = \sqrt{13}(\sin(\phi)\sin(x) + \cos(\phi)\cos(x)) = 1$$
$$\sin(\phi + x) = \frac{1}{\sqrt{13}}$$
$$\phi = \arcsin\left(\frac{2}{\sqrt{13}}\right) \therefore \sin\left(\arcsin\left(\frac{2}{\sqrt{13}}\right) + x\right) = \frac{1}{\sqrt{13}}$$

The general method of solution can be used from here.

Finding Trigonometric Function Values

Example

If $\tan x = 2, x \in [0, \frac{\pi}{2}]$, find $\sec(x), \cos(x), \sin(x), \text{cosec}(x)$

$\tan x = 2, \quad x \in [0, \pi/2]$

$\sec x$: $1 + \tan^2 x = \sec^2 x \quad \therefore \sec^2 x = 2^2 + 1 = 5$

$\therefore \sec x = \pm\sqrt{5} \quad \because x \in [0, \frac{\pi}{2}] \quad \therefore \sec x > 0.$

$\therefore \sec x = \sqrt{5}$

$\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{5}}$

$\sin x = \pm\sqrt{1 - \cos^2 x} = \pm\sqrt{1 - \frac{1}{5}} = \pm\sqrt{\frac{4}{5}} = \pm\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \quad (\because \sin x > 0)$

$\text{cosec}\, x = \frac{1}{\sin x} = \frac{\sqrt{5}}{2}$

if $\sin(x) = -\frac{1}{5}, \pi \le x \le \frac{3\pi}{2}$, find $\cot(x)$.

$1 + \cot^2 x = \text{cosec}^2 x \quad \therefore \cot^2 x = \text{cosec}^2 x - 1, \quad \text{cosec}\, x = -5$

$\therefore \cot^2 x = (-5)^2 - 1 = 25 - 1 = 24$

$\therefore \cot x = \pm\sqrt{24} = \pm 2\sqrt{6}. \quad \because \pi \le x \le \frac{3\pi}{2} \quad \therefore \cot x \ge 0$

$\therefore \cot x = 2\sqrt{6}.$

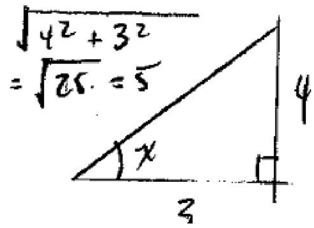
Solution using Right-Angled Triangle

Warning: this method works best for angles in Q1. Outside of Q1, a different approach needs to be taken in order for the method to be mathematically sound.

Example 1:

Find $\cos(x)$ if $\tan(x) = \frac{4}{3}$, $0 \le x \le \frac{\pi}{2}$

E12. $\tan x = \frac{4}{3}$, $x \in [0, \frac{\pi}{2}]$

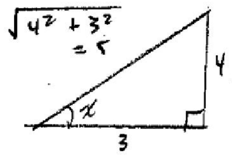


$\therefore \cos x = \frac{A}{H} = \frac{3}{5}$

Example 2:

Find $\cos(x)$ if $\tan(x) = \frac{4}{3}$, $\pi \le x \le \frac{3\pi}{2}$

E13. $\tan x = \frac{4}{3}$, $x \in [\pi, \frac{3\pi}{2}]$. INCORRECT SOLUTION.



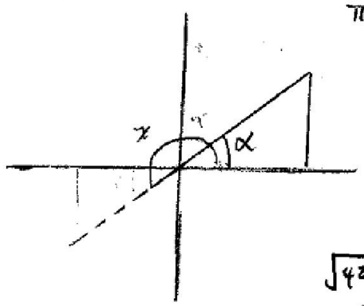
$\therefore \cos x = \frac{A}{H} = \frac{3}{5}$

$\because x \in [\pi, \frac{3\pi}{2}]$ $\therefore \cos x < 0$.

$\therefore \cos x = -\frac{3}{5}$

★ Incorrect solution.

Proper method:

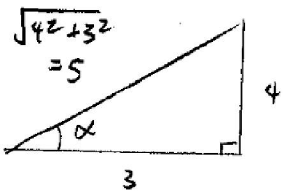


$\pi \le x \le \frac{3\pi}{2}$

Let $\alpha = x - \pi$, $\alpha \in [0, \frac{\pi}{2}]$

$\tan x = \tan(\alpha + \pi) = \tan \alpha = \frac{4}{3}$

$\therefore \tan \alpha = \frac{4}{3}$



$\therefore \cos \alpha = \frac{A}{H} = \frac{3}{5}$

$\therefore \cos x = \cos(\alpha + \pi) = -\cos \alpha$

$= -\frac{3}{5}$

Complex numbers

The set of complex numbers is denoted by $\mathbb{C}$, $\mathbb{C} = \{z \colon z = x + yi, x, y \in \mathbb{R}\}$.

Complex numbers are typically denoted z, $z = x + yi$, where $i = \sqrt{-1}$.

For any complex number $z = x + yi$:

- $Re(z) = x$
- $Im(z) = y$

Properties of Complex Numbers

- Equality: $a + bi = c + di \Leftrightarrow a = c \wedge b = d$
- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$
- Negation: $-(a + bi) = (-a) + (-b)i$
- Zero: the zero element of $\mathbb{C}$ is $0 + 0i$
- Real Multiplication: $k(a + bi) = ka + kbi$
- Complex Multiplication: $(a + bi)(c + di) = ac + adi + bci + bdi^2$
  $= ac - bd + (bc + ad)i$
- Conjugates: If $z = a + bi$, then the conjugate of z, denoted $\bar{z}$, is given as $\bar{z} = a - bi$

Properties of the Conjugate

- $z + \bar{z} = 2\,\mathrm{Re}(z)$
- $z - \bar{z} = 2\,\mathrm{Im}(z)\,i$
- $z\bar{z} = |z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$
- $\bar{\bar{z}} = z$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Example

If $z = 2i$, find $|z^2|$ and $\mathrm{Arg}(z^2)$.

$z = 2i \quad \therefore z^2 = 4i^2 = -4 \qquad \therefore |z^2| = 4, \quad \mathrm{Arg}(z^2) = \pi.$

Important: the conjugate can be very useful in simplifying expressions involving complex numbers. For example:

If $u = z + \frac{1}{z}$ where $|z| = 1$, find Im(u)

$$u = z + \frac{1}{z} = z + \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = z + \frac{\bar{z}}{|z|^2} = z + \bar{z} \Rightarrow \mathrm{Im}(u) = 0$$

This could also be done using the $a + bi$ form, but it is much faster.

Division of Complex Numbers

Division can be carried out by a process similar to rationalisation of the denominator. For a fraction $\frac{a+bi}{c+di}$, multiply by $\frac{c-di}{c-di}$ to make the denominator real.

Example:

$$\frac{2-3i}{4-i}\cdot\frac{4+i}{4+i} = \frac{8+3-12i+2i}{4^2+1} = \frac{11-10i}{17} = \frac{11}{17}-\frac{10}{17}i$$

Order – $\mathbb{C}$ is not ordered, i.e. it is not meaningful to say $z_1 > z_2$

Modulus and Argument

The **modulus** of z, $|z|$, is given by $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$

The **argument** of z, Arg or arg, is given as:

- $Arg(z)$ is restricted to the domain $(-\pi, \pi]$
- $\arg(z)$ is unrestricted, that is, it has the domain $\mathbb{R}$

$\mathrm{Arg}(z)$ can be found using: $\mathrm{Arg}(z) = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) \pm \pi$

- Add nothing if z is in Q1 or Q4
- Add π if z is in Q2
- Subtract π if z is in Q3

Properties of Modulus and Argument

- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $|z^n| = |z|^n$
- $|z_1| = |z_2| \Leftrightarrow |uz_1| = |uz_2|, u \in \mathbb{C}$
- $|z| = |\bar{z}|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- $\arg(\bar{z}) = -\arg(z)$

Example:

Find $|z|$ and $\mathrm{Arg}(z)$ if $z = 3+4i$

$|z| = \sqrt{9+16} = \sqrt{25} = 5$

$\mathrm{Arg}(z) = \tan^{-1}\left(\frac{4}{3}\right)$

Polar Form

The polar form of a complex number is

$$z = r(\cos(\theta) + i\sin(\theta)) = r\,\mathrm{cis}(\theta)$$

r is the modulus of z, and θ is the argument of z

Multiplication

$$r_1 \operatorname{cis}(\theta_1) \cdot r_2 \operatorname{cis}(\theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Division

$$\frac{r_1 \operatorname{cis}(\theta_1)}{r_2 \operatorname{cis}(\theta_2)} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

De Moivre's Theorem

De Moivre's Theorem gives a rule for finding powers of a complex number in polar form.

$$[r \operatorname{cis}(\theta)]^n = r^n \operatorname{cis}(n\theta)\,, n \in \mathbb{R}$$

Example:

Find the polar form of $\sqrt{3}i - 1$.

Let $z = -1 + \sqrt{3}i$ $\quad \therefore |z| = \sqrt{1+3} = \sqrt{4} = 2$

$\operatorname{Arg}(z) = \tan^{-1}(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$ $\quad \therefore z = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$

If $z = 2i$, find $|z^2|$ and $\operatorname{Arg}(z^2)$.

$z = 2i$ $\quad \therefore z^2 = 4i^2 = -4$ $\quad \therefore |z^2| = 4$, $\operatorname{Arg}(z^2) = \pi$.

Let $z_1 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

(a) Express z_1 in polar form.

$|z_1| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$ $\quad Arg(z_1) = \tan^{-1}\left(\frac{-1/\sqrt{2}}{1/\sqrt{2}}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\therefore z_1 = cis\left(-\frac{\pi}{4}\right)$

(b) $z_2 = cis\left(\frac{\pi}{3}\right)$. Find $z_1 z_2$ in polar form.

$z_1 z_2 = cis\left(-\frac{\pi}{4}\right) cis\left(\frac{\pi}{3}\right) = cis\left(-\frac{\pi}{4} + \frac{\pi}{3}\right) = cis\,\frac{\pi}{12}$

(c) (i) Find $z_1 z_2$ in cartesian form.

$z_1 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \quad z_2 = cis\,\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore z_1 z_2 = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}i + \frac{\sqrt{3}}{2\sqrt{2}}i$

$= \frac{1+\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}}i$

(ii) Hence find $\sin\frac{\pi}{12}$.

$\sin\frac{\pi}{12} = Im\left(cis\,\frac{\pi}{12}\right) = Im(z_1 z_2) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(d) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Let $w = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

(a) Find w in polar form

$|w| = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = 1.$ $\quad Arg(w) = \tan^{-1}\left(\frac{-1/2}{-\sqrt{3}/2}\right) - \pi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \pi$

$= \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

$\therefore w = cis\left(-\frac{5\pi}{6}\right)$

(b) Hence find the least positive integer k such that $w^k = 1$.

$w^k = cis\left(-\frac{5\pi}{6} \times k\right) = 1 = cis(2n\pi)$

$\therefore -\frac{5k\pi}{6} = \pm 2n\pi \Rightarrow k = \frac{\pm 12n\pi}{5\pi} = \frac{\pm 12n}{5} \quad \therefore$ when $n = 5$, $k = 12$

Factorisation of Polynomials in $\mathbb{C}$

Fundamental Theorem of Algebra

Every expression $P(z) = a_n z^n + a_{(n-1)} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0, a_n \neq 0$, where n is a natural number and $a_0, \dots a_n \in \mathbb{C}$ has at least one linear factor in the complex number system.

I.e. any polynomial with complex coefficients of degree n has n linear factors in $\mathbb{C}$.

Conjugate Factor Theorem

If $P(z)$ has real coefficients, and $P(\alpha) = 0$ where $\alpha \in \mathbb{C}$, then $P(\bar{\alpha}) = 0$, i.e. if $(z - \alpha)$ is a factor, then $(z - \bar{\alpha})$.

Sum of Perfect Squares

$$a^2 + b^2 = (a + bi)(a - bi)$$

Complex Factorisation of Quadratics

- Complete the square
- Use sum of perfect squares to factorise.

Example: factor $z^2 + 3z + 4$

$$\begin{aligned} z^2 + 3z + 4 &= z^2 + 3z + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4 \\ &= \left(z + \frac{3}{2}\right)^2 - \frac{9}{4} + 4 = \left(z + \frac{3}{2}\right)^2 + \frac{7}{4} \\ &= \left(z + \frac{3}{2} + \frac{\sqrt{7}}{2} i\right)\left(z + \frac{3}{2} - \frac{\sqrt{7}}{2} i\right) \end{aligned}$$

Complex Factorisation of Higher Degree Polynomials

- Use factor theorem to find linear factors, polynomial long division to reduce to linear factors and a quadratic
- Complex factor the quadratic

Example

Find all the linear factors of $z^6 - 27$.

$z^6 - 27 = 0 \Rightarrow z^6 = 27 = 27\operatorname{cis}(0 + 2k\pi),\ k \in \mathbb{Z}$.

$\therefore z = [27\operatorname{cis}(2k\pi)]^{\frac{1}{6}}$, interval $= \frac{2\pi}{6} = \frac{\pi}{3}$

$\therefore z = 27^{\frac{1}{6}}\operatorname{cis}(2k\pi) = (3^3)^{\frac{1}{6}}\operatorname{cis}(2k\pi) = \sqrt{3}\operatorname{cis}(2k\pi)$

$\therefore z = \sqrt{3}\operatorname{cis}\left(\frac{\pi}{3}\right),\ \sqrt{3}\operatorname{cis}\left(\frac{2\pi}{3}\right),\ \sqrt{3}\operatorname{cis}(\pi),\ \sqrt{3}\operatorname{cis}\left(-\frac{\pi}{3}\right),\ \sqrt{3}\operatorname{cis}\left(-\frac{2\pi}{3}\right),\ \sqrt{3}\operatorname{cis}(0)$

$= \sqrt{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),\ \sqrt{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),\ -\sqrt{3},\ \sqrt{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right),\ \sqrt{3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right),\ \sqrt{3}$

$= \frac{\sqrt{3}}{2} + \frac{3}{2}i,\ -\frac{\sqrt{3}}{2} + \frac{3}{2}i,\ -\sqrt{3},\ \frac{\sqrt{3}}{2} - \frac{3}{2}i,\ -\frac{\sqrt{3}}{2} - \frac{3}{2}i,\ \sqrt{3}$

$\therefore z^6 - 27 = \left(z - \frac{\sqrt{3}}{2} - \frac{3}{2}i\right)\left(z + \frac{\sqrt{3}}{2} - \frac{3}{2}i\right)(z + \sqrt{3})\left(z - \frac{\sqrt{3}}{2} + \frac{3}{2}i\right)\left(z + \frac{\sqrt{3}}{2} + \frac{3}{2}i\right)(z - \sqrt{3})$

Solve $z^3 - z^2 + 2z - 8 = 0$ over $\mathbb{C}$, given $(z-2)$ a factor

$P(z) = z^3 - z^2 + 2z - 8$, $P(2) = 8 - 4 + 4 - 8 = 0$.

$$\begin{array}{r l}
 & z^2 + z + 4 \\
z-2 & \overline{)\,z^3 - z^2 + 2z - 8} \\
 & \underline{-z^3 + 2z^2} \\
 & z^2 + 2z \\
 & \underline{-z^2 + 2z} \\
 & 4z - 8 \\
 & \underline{4z - 8} \\
 & 0
\end{array}$$

$\therefore P(z) = (z-2)(z^2 + z + 4)$

$= (z-2)\left(z^2 + z + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 4\right)$

$= (z-2)\left[\left(z + \frac{1}{2}\right)^2 - \frac{1}{4} + 4\right]$

$= (z-2)\left[\left(z + \frac{1}{2}\right)^2 + \frac{15}{4}\right]$

$= (z-2)\left(z + \frac{1}{2} + \frac{\sqrt{15}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{15}}{2}i\right) = 0$.

$\therefore z = 2,\ -\frac{1}{2} - \frac{\sqrt{15}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{15}}{2}i$ //

Solve $z^3 - (2-i)z^2 + z - 2 + i = 0$.

$P(z) = z^3 - (2-i)z^2 + z - 2 + i = z^2(z - (2-i)) + (z - (2-i))$

$= (z - (2-i))[z^2 + 1] = (z - (2-i))(z - i)(z + i) = 0$.

$\therefore z = 2 - i,\ \pm i$ //

Given that $(z-2i)$ is a factor, factorize $z^3-(2+2i)z^2-(3-4i)z+6i$ completely

$$P(z) = z^3-(2+2i)z^2-(3-4i)z+6i$$

$$\begin{array}{r l}
 & z^2 - 2z - 3 \\
z-2i \,\big) & z^3+(2+2i)z^2-(3-4i)z+6i \\
 & -\left(z^3 - 2iz^2\right) \\
 & -2z^2-(3-4i)z \\
 & +2z^2 - 4iz \\
 & -3z+6i \\
 & -3z+6i \\
 & 0
\end{array}$$

$$\therefore P(z) = (z-2i)(z^2-2z-3)$$
$$= (z-2i)(z-3)(z+1)$$

Given $1-i$ is a solution of $z^3-4z^2+6z-4=0$, find the other two roots.

- By the Conjugate Factor Theorem, $\overline{1-i} = 1+i$ also a factor

$$(z-(1-i))(z-(1+i)) = (z-1+i)(z-1-i) = (z-1)^2-i^2 = (z-1)^2+1.$$
$$= z^2+2z+1+1$$
$$= z^2-2z+2 \quad \text{a factor}$$

$$\begin{array}{r l}
 & z - 2 \\
z^2-2z+2 \,\big) & z^3-4z^2+6z-4 \\
 & -\left(z^3-2z^2+2z\right) \\
 & -2z^2+4z-4 \\
 & -2z^2+4z-4 \\
 & 0
\end{array}$$

$\therefore z-2$ the remaining factor

$\therefore$ Other roots are:

$z = 1+i,\ 2$

Solve x^2-4x+5 over $\mathbb{C}$:

$$x = \frac{4\pm\sqrt{16-4(5)}}{2} = \frac{4\pm\sqrt{-4}}{2} = \frac{4\pm 2i}{2} = 2\pm i$$

Completing the square can also be done for non-quadratics on some cases:

Solve $z^4 - 8z^2 + 49 = 0$

$$\begin{aligned}
&z^4 - 8z^2 + 49 = z^4 - 14z^2 + 49 + 6z^2 = 0 \\
&(z^2 - 7)^2 + 6z^2 = 0 \\
&(z^2 - 7)^2 - (\sqrt{6}iz)^2 = 0 \\
&(z^2 - 7 - \sqrt{6}iz)(z^2 - 7 + \sqrt{6}iz) = 0
\end{aligned}$$

z can now be solved for using the quadratic formula.

Solution of Equations of the Form $z^n = a$

- There will be n solutions.
- Each solution will be spaced at intervals of $2\pi/n$ around the origin
- Each solution has a modulus of $|a|^{\frac{1}{n}}$

Method 1 – De Moivre's Theorem

$$\text{Let } a = r\operatorname{cis}(\theta)$$

$$\therefore z^n = r\operatorname{cis}(\theta) \Rightarrow z = (r\operatorname{cis}(\theta))^{\frac{1}{n}} = r^{\frac{1}{n}}\operatorname{cis}\left(\frac{1}{n}(\theta)\right)$$

$$z = r^{\frac{1}{n}}\operatorname{cis}\left(\frac{1}{n}(\theta + 2k\pi)\right), k \in \mathbb{Z}$$

Now all we need to do is let k=0, then find one solution $z = r^{\frac{1}{n}}\operatorname{cis}(\theta/n)$ and then find the other solutions from this by adding or subtracting the interval $2\pi/n$, making sure that the Argument remains in the interval $(-\pi, \pi]$.

Method 2 – Cartesian Form

This method only works if $n = 2$. The method is to let $z = a + bi$, expand z^n and solve equations for x and y. **Not the preferred method**.

Example:

Find cube roots of $-2+2i$ in polar form.

$|-2+2i| = \sqrt{4+4} = 2\sqrt{2}$, $\operatorname{Arg}(-2+2i) = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$\therefore -2+2i = 2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

$\therefore \left[2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)\right]^{\frac{1}{3}} = \left(2\sqrt{2}\right)^{\frac{1}{3}}\operatorname{cis}\left(\frac{1}{3}\left(\frac{3\pi}{4} + 2k\pi\right)\right)$, $k \in \mathbb{Z}$.

Interval $= \frac{2\pi}{3} = \frac{8\pi}{12}$

$\therefore$ Cube roots $= \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}}\operatorname{cis}\left(\frac{1}{3}\left(\frac{3\pi}{4} + 2k\pi\right)\right) = 2^{\frac{1}{2}}\operatorname{cis}\left(\frac{1}{3}\left(\frac{3\pi}{4} + 2k\pi\right)\right)$

$\Rightarrow \sqrt{2}\operatorname{cis}\left(\frac{3\pi}{12}\right), \sqrt{2}\operatorname{cis}\left(\frac{11\pi}{12}\right), \sqrt{2}\operatorname{cis}\left(-\frac{5\pi}{12}\right)$

$\Rightarrow \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right), \sqrt{2}\operatorname{cis}\left(\frac{11\pi}{12}\right), \sqrt{2}\operatorname{cis}\left(-\frac{5\pi}{12}\right)$

Find the 2001st roots of unity

$1 = cis(0)$. $\therefore 1^{\frac{1}{2001}} = [cis(2n\pi)]^{\frac{1}{2001}} = cis\left(\frac{1}{2001}(2n\pi)\right)$, $n \in$

$\because Arg(z) \in (-\pi, \pi]$. $\therefore \frac{1}{2001}(2n\pi) \in (-\pi, \pi]$

$\therefore 2n\pi \in (-2001\pi, 2001\pi]$

$2n \in (-2001, 2001]$

$\therefore n \in \left(\frac{-2001}{2}, \frac{2001}{2}\right]$, $n \in \mathbb{Z}$.

Find the cube roots of $8i$.

$z^3 = 8i$ $\therefore z^3 = 8cis\left(\frac{\pi}{2}\right)$ $\therefore z = \left[8cis\left(\frac{\pi}{2} + 2k\pi\right)\right]^{\frac{1}{3}}$, $k \in \mathbb{Z}$.

$\therefore z = 2cis\left(\frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right)\right)$ interval $= \frac{2\pi}{3} = \frac{4\pi}{6}$

$= 2cis\left(\frac{\pi}{6}\right), 2cis\left(\frac{5\pi}{6}\right), 2cis\left(-\frac{\pi}{2}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), -2i$

$= \sqrt{3} + i, -\sqrt{3} + i, -2i$

<u>**Relations and Regions of the Complex Plane**</u>

These relations are given in set notation, and refer to a set of points in $\mathbb{C}$ which can be represented on the Argand diagram. For example, $\{z\colon \mathrm{Re}(z) = \mathrm{Im}(z)^2 + 2\,\mathrm{Im}(z) + 2, z \in \mathbb{C}\}$.

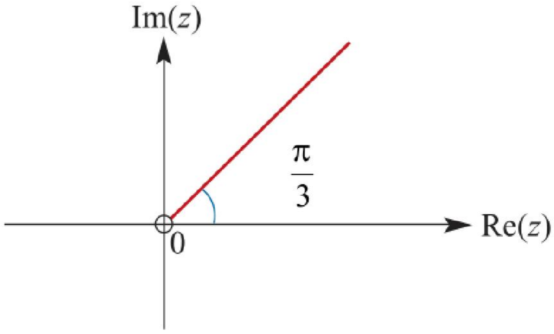
There are two main ways of working these out:

- **Recognition** – there are certain forms of relations which can be identified and interpreted geometrically. Works well for 'simple' relations.
- Cartesian Substitution – substitution of $z = x + yi$ in order to yield an relation between x and y which can then be sketched. Works well for relations which are more complex, and as a last resort.

Important - $|z - a|$ is interpreted as giving the **distance** between z and a**,** i.e. *the length of the line segment joining z and a on the Argand diagram.* (Think of complex numbers here as two-dimensional vectors, with $\mathrm{Re}(z)$ as the i-component, and $\mathrm{Im}(z)$ as the j-component)

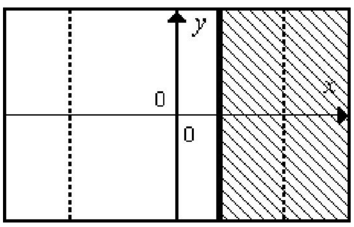
Common Relations to be Recognized

- $|z - a| = |z - b|$ - the perpendicular bisector of the line segment joining a and b, a geometric result.
- $|z - a| = r$ – a circle centre a with radius r.
- Any equation of the form $|z - a| = k|z - b|$, $k \in \mathbb{R}^+, k \neq 1$ is a *circle.*
- $\mathrm{Arg}(z - a) = \phi$ – a ray centred at a, making an angle ϕ with the positive $\mathrm{Re}(z)$ axis. **Important** – $\mathrm{Arg}(0)$ is undefined, hence there is a **HOLE** at $z = a$.
  Example: $\{z\colon \mathrm{Arg}(z) = \frac{\pi}{3}\}$

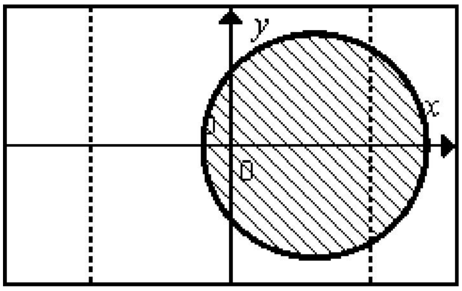


Examples

$\{z\colon \mathrm{Re}(z) \geq 2\}$ represents $x \geq 2$, i.e.

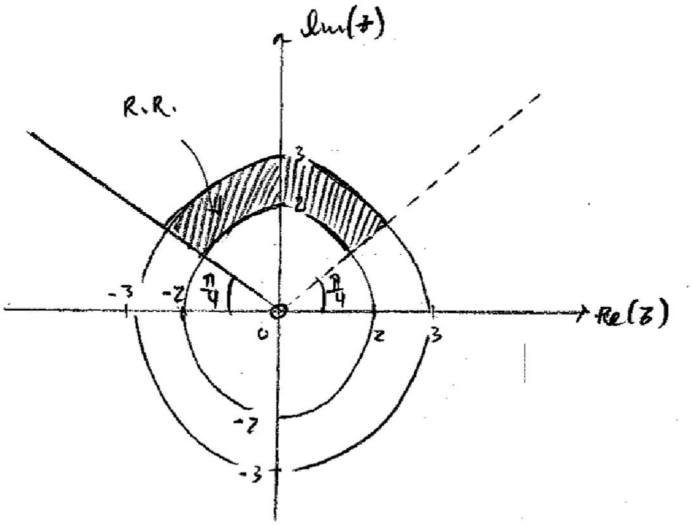


$\{z\colon |z - 3| \leq 4\}$ represents the region within a circle centre $3 + 0i$ with radius 4, i.e.



Examples:

Sketch $\{z: 2 \le z \le 3\} \cap \{z: \frac{\pi}{4} < Arg(z) \le \frac{3\pi}{4}\}$.



$C = \{z: \left|\frac{z-\bar{z}}{z+\bar{z}}\right| \le 1\}$, $D = \{z: z^2 + (\bar{z})^2 \le 2\}$. sketch $C \cap D$.

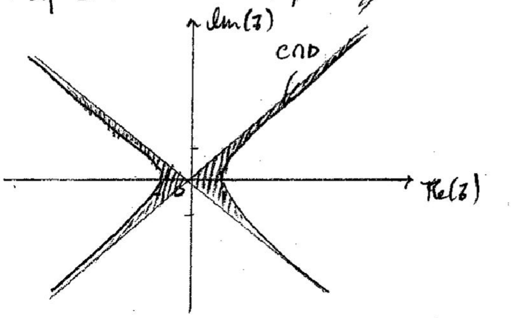
C: $\left|\frac{z-\bar{z}}{z+\bar{z}}\right| = \left|\frac{2Im(z)i}{2Re(z)}\right|$, let $z = x+yi$ ∴ $\left|\frac{2Im(z)i}{2Re(z)}\right| = \left|\frac{2yi}{2x}\right| \le 1$.

$\left|\frac{y}{x}i\right| \le 1 \Rightarrow \left|\frac{y}{x}\right| \le 1 \Rightarrow |y| \le |x| \Rightarrow y^2 \le x^2 \Rightarrow y^2 - x^2 \le 0$.

D: $z^2 + (\bar{z})^2 \le 2$. Let $z = x+yi$

$x^2 - y^2 + 2xyi + \overline{(x^2 - y^2 + 2xyi)} = x^2 - y^2 + \cancel{2xyi} + x^2 - y^2 - \cancel{2xyi}$

$= 2x^2 - 2y^2 \le 2 \Rightarrow x^2 - y^2 \le 1$



Sketch in the Argand Diagram:

(a) $|z-1|+|z+1| = 3.$

Let $z = x+yi.$

$\therefore |(x-1)+yi| + |(x+1)+yi| = 3$

$\therefore \sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} = 3$

$\sqrt{(x-1)^2+y^2} = 3 - \sqrt{(x+1)^2+y^2}$

$(x-1)^2+\cancel{y^2} = 9 - 6\sqrt{(x+1)^2+y^2} + (x+1)^2 + \cancel{y^2}$

$\cancel{x^2} - 2x + \cancel{1} = 9 - 6\sqrt{(x+1)^2+y^2} + \cancel{x^2} + 2x + \cancel{1}$

$6\sqrt{(x+1)^2+y^2} = 4x+9.$

$36\left[(x+1)^2+y^2\right] = (4x+9)^2$

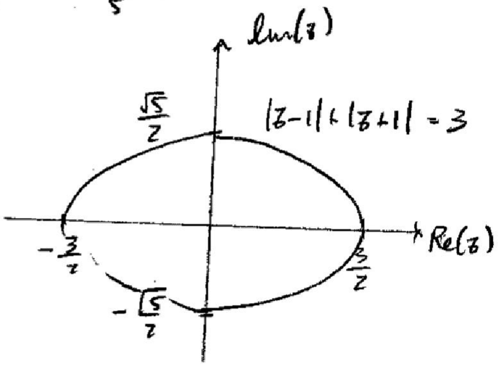
$36(x^2+2x+1)+36y^2 = 16x^2+72x+81$

$36x^2+\cancel{72x}+36+36y^2 = 16x^2+\cancel{72x}+81$

$20x^2+36y^2 = 81-36 = 45$

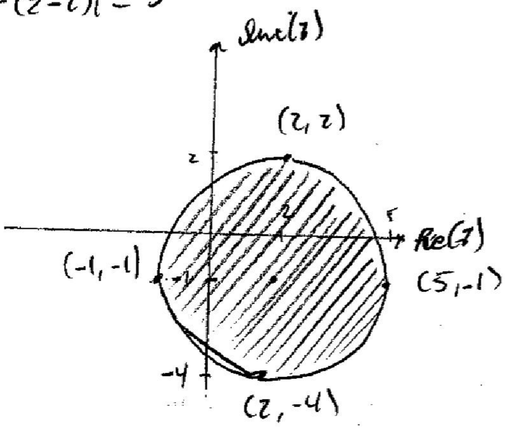
$$\frac{20x^2}{45} + \frac{36y^2}{45} = \frac{4x^2}{9} + \frac{4y^2}{5} = 1.$$

$$\therefore \frac{x^2}{\left(\frac{3}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt5}{2}\right)^2} = 1.$$



(b) $|z-2+i| \le 3$

$\therefore |z-(2-i)| \le 3$



Differential calculus

Derivatives of Standard Functions

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln|x| = \frac{1}{x}$$

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\operatorname{cosec}(x) = -\operatorname{cosec}(x)\cot(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\cot(x) = -\operatorname{cosec}^2(x)$$

$$\frac{d}{dx}\arcsin\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}} \qquad \frac{d}{dx}\arccos\left(\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2-x^2}} \qquad \frac{d}{dx}\arctan\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$$

$$\frac{d}{dx}|x| = \begin{cases} 1, x > 0 \\ -1, x < 0 \end{cases}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln(a)} \text{ (!)} \qquad \frac{d}{dx}a^x = \ln(a)\, a^x \text{ (!)}$$

(!) – not in the course of SM, make sure you **derive from scratch** if using in short answer/extended response

Differentiation Rules

$$\frac{d}{dx}(uv) = u'v + v'u \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}$$

Examples

If $y = 3\tan^{-1}(w)$, where w is a function of x, then $\frac{dy}{dx} = ?$

$$\frac{dy}{dx} = 3\frac{d}{dx}\tan^{-1}(w) = 3\frac{dw}{dx}\cdot\frac{1}{1+w^2} = \frac{3}{1+w^2}\cdot\frac{dw}{dx}$$

If $y = 2\ln(3x)$, find $y''(x)$.

$$\frac{dy}{dx} = 2\cdot\cancel{3}\cdot\frac{1}{\cancel{3}x} = \frac{2}{x} \quad \therefore \frac{d^2y}{dx^2} = 2\cdot\frac{-1}{x^2} = \frac{-2}{x^2}$$

If $y = \frac{1-\sin x}{\cos x}$, show that $\frac{dy}{dx} = -\frac{1}{1+\sin x}$, and find $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\cos x} - \tan x\right) = \frac{d}{dx}(\sec x - \tan x) = \sec x \tan x - \sec^2 x$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} - \frac{1}{\cos^2 x} = \frac{\sin x - 1}{1 - \sin^2 x} = -\frac{\cancel{1-\sin x}}{(\cancel{1-\sin x})(1+\sin x)}$$

$$= -\frac{1}{1+\sin x}$$

$$\frac{d^2y}{dx^2} = +\frac{\cos x}{(1+\sin x)^2}$$

Limit Properties

- $\lim_{x\to a} f(x)$ exists if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$
$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$$

$$\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$$
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{(x\to c)} g(x)}$$

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$$

[Provided f(x) continuous]

L'Hopital's Rule

Applies **only** to the case where simple limit evaluation results in indeterminate form, i.e. $\pm\frac{\infty}{\infty}, \pm\frac{0}{0}$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Continuity

A function $f(x)$ is continuous at $x = a$ if:

- $f(a)$ exists
- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$

Differentiability

A function $f(x)$ is differentiable at $x = a$ if:

- $f(x)$ is continuous at $x = a$
- $\lim_{x\to a} f'(x)$ exists

As a result, the derivative of a function is **undefined** at its endpoints ($\lim_{x\to a} f'(x)$ does not exist, and function is not continuous)

Note : differentiability implies continuity, but the converse is not true. Consider $y = |x|$.

Implicit Differentiation

Differentiation of an implicit relation can be done via the chain rule, since $\frac{d}{dx}\big(f(y)\big) = \frac{df}{dy} \cdot \frac{dy}{dx}$ where y is related to x (not necessarily a function, could be any relation)

Quick rule – if you need to differentiate an expression in terms of y, differentiate it *dy* first and then multiply $\frac{dy}{dx}$ at the end, e.g. $\frac{d}{dx}(y^2 + \sin(y)) = (2y + \cos(y)) \cdot \frac{dy}{dx}$

Example

Find $\frac{dy}{dx}$ if:

(a) $x\cos y + y\cos x = 1$

(b) $y = \sin(x+y)$

(a) $\cos y + (-\sin y)\frac{dy}{dx} \cdot x + \frac{dy}{dx}\cos x + (-\sin x)\, y = 0.$

$-x\sin y \frac{dy}{dx} + \frac{dy}{dx}\cos x = -\cos y + y\sin x$

$\frac{dy}{dx} = \frac{y\sin x - \cos y}{\cos x - x\sin y}$

(b) $y = \sin(x+y) \quad \therefore \quad \frac{dy}{dx} = \left(1 + \frac{dy}{dx}\right)\cos(x+y) = \cos(x+y) + \frac{dy}{dx}\cos(x+y)$

$\therefore \frac{dy}{dx} - \frac{dy}{dx}\cos(x+y) = \cos(x+y)$

$\therefore \frac{dy}{dx} = \frac{\cos(x+y)}{1-\cos(x+y)}$

Find the gradient of the circle $x^2+y^2=1$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\frac{dy}{dx}$: $\quad 2x + 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{-2x}{+2y} = -\frac{x}{y}$

$\frac{dy}{dx}\Big|_{\left(\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right)} = \frac{-1/\sqrt{2}}{+1/\sqrt{2}} = -1$

If $u = 2f(x)$, $x = 3h(t)$, find $\frac{d^2u}{dt^2}$.

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = 2\frac{df}{dx} \cdot 3\frac{dh}{dt} = 6\frac{df}{dx}\frac{dh}{dt}$$

$$\frac{d^2u}{dt^2} = 6\left[\frac{d}{dt}\left(\frac{df}{dx}\right) \cdot \frac{dh}{dt} + \frac{d}{dt}\left(\frac{dh}{dt}\right) \cdot \frac{df}{dx}\right]$$

$$= 6\left[\frac{d^2f}{dx^2} \cdot \frac{dx}{dt} \cdot \frac{dh}{dt} + \frac{d^2h}{dt^2} \cdot \frac{df}{dx}\right]$$

$$= 6\left(\frac{d^2f}{dx^2} \cdot 3\frac{dh}{dt} \cdot \frac{dh}{dt} + \frac{d^2h}{dt^2} \cdot \frac{df}{dx}\right)$$

$$= 18\frac{d^2f}{dx^2}\left(\frac{dh}{dt}\right)^2 + 6\frac{d^2h}{dt^2} \cdot \frac{df}{dx}$$

Logarithmic Differentiation

Differentiating some expressions can be made easier by taking ln(...) of both sides, and then implicitly differentiating both sides.

Example: Differentiate $y = [\sin(x)]^{e^x}$

$$y = [\sin(x)]^{e^x} \Rightarrow \ln(y) = e^x \ln[sin(x)] \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = e^x \ln[\sin(x)] + \frac{\cos(x)}{\sin(x)} e^x$$

$$\frac{dy}{dx} = y[e^x \ln(\sin(x)) + \cot(x)\, e^x] = (\sin(x))^{e^x} [e^x \ln(\sin(x)) + \cot(x)\, e^x]$$

Parametric Equations

To differentiate a parametric equation, one can find $\frac{dy}{dx}$ by using the formula $\frac{dy}{dx} = (\frac{dy}{dt})/(\frac{dx}{dt})$.

Example

Find the gradient of $\begin{cases} x = t^2+1 \\ y = \sin t \end{cases}$ at any time t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{dy}{dt} = \cos t, \qquad \frac{dx}{dt} = 2t.$$

$$\therefore \frac{dy}{dx} = \frac{\cos t}{2t}.$$

Applications of Differentiation

Angles between Lines

The angle θ between two lines at a point of intersection is given as $|\theta_2 - \theta_1|$, where θ_2 and θ_1 are the angles made with the positive x-axis by each of the lines. It can be shown that:

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$$

Where α is the angle between the lines, and m_1, m_2 are the gradients of each line.

Angles between Curves

The angle between two curves at a point of intersection is defined as the angle between the tangents to each curve. In this case, m_2, m_1 simply become the gradients of each curve at the specified point, and the above equation is used.

Example

Find the angle between the curves $f(x) = 4 - x^2$ and $g(x) = x^2 - 4$ at $(2,0)$.

$f'(x) = -2x$, $g'(x) = 2x$., $f'(2) = -4$, $g'(2) = 4$.

$$\tan\theta = \frac{|m_2 - m_1|}{|1 + m_1 m_2|} = \frac{|4-(-4)|}{|1+(-4)(4)|} = \frac{8}{|1-16|} = \frac{8}{15}$$

$$\therefore \theta = \tan^{-1}\left(\frac{8}{15}\right) = 28.07^\circ$$

Linear Approximation

$$\delta y \approx \frac{dy}{dx}\delta x \text{ or } f(x+h) \approx f(x) + hf'(x), \text{where } h \text{ is small}$$

Example

If $f(x) = 4x^3 + 3x^2 + 9x + 10$, find an approximate value for $f(4.01)$.

$f'(x) = 12x^2 + 6x + 9$.

$$f(4.01) \approx f(4) + 0.01 f'(4).$$
$$= 4(4^3) + 3(4^2) + 9(4) + 10 + 0.01\left(12(4^2) + 6(4) + 9\right)$$
$$= 352.25$$

Percentage Change

$$\%(\text{change}) = 100\left(\frac{f(a+h) - f(a)}{f(a)}\right) \approx \frac{100hf'(a)}{f(a)} = \frac{100\delta y}{y}$$

Example

The area of a circular disc increases from 100π cm^2 to 101π cm^2. Find an approximation of the corresponding increase in radius.

$\delta r \approx \frac{dr}{dA}\delta A$, $A = \pi r^2$ $\therefore \frac{dA}{dr} = 2\pi r$ $\Rightarrow \frac{dr}{dA} = \frac{1}{2\pi r}$

$\therefore \delta r \approx \frac{1}{2\pi r}\delta A$. $A = 100\pi = \pi r^2 \Rightarrow r = 10$.

$\delta A = 101\pi - 100\pi = \pi$.

$\therefore \delta r \approx \frac{1}{2\cancel{\pi}(10)} \times \cancel{\pi} = \frac{1}{20}$ cm

Concavity

Concavity can be loosely defined as whether a graph "holds water" (concave up) or "pours water" (concave down).

- A graph is concave up at (x, y) if d^2y/dx^2 is positive
- A graph is concave down at (x, y) if d^2y/dx^2 is negative
- A graph has **zero concavity** at (x, y) if d^2y/dx^2 is zero

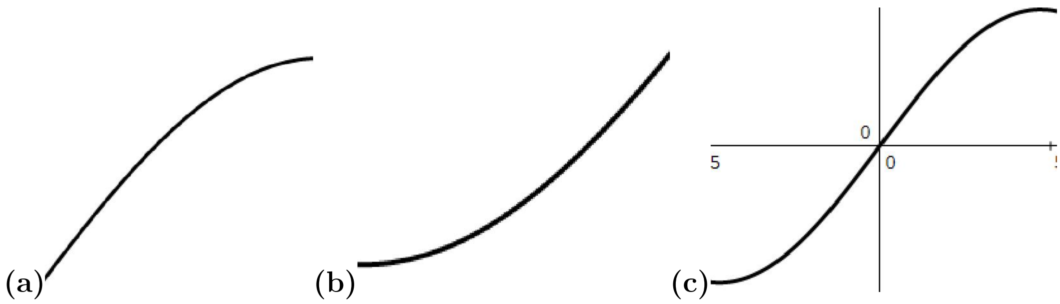


Figure 2 - Concavity. (a) - concave down, (b) - concave up, (c) zero concavity at x=0

Stationary Points

Points on a graph where $dy/dx = 0$ are called *stationary points*. For univariate calculus, stationary points are classified as:

- **Local minimum**
- **Local maximum**
- **Stationary point of inflection**

Local Minimum

A stationary point is a local minimum if dy/dx changes from negative to positive at that point, or if $\frac{d^2y}{dx^2} > 0$ at the point.

Local Maximum

A stationary point is a local maximum if dy/dx changes from positive to negative at that point, or if $\frac{d^2y}{dx^2} < 0$ at the point.

Points of Inflection

A point is a point of inflection if dy/dx does *not change sign* at the point, i.e. it is a local minimum/maximum of dy/dx, or if d^2y/dx^2 changes sign around the point. **A point of inflection IS NOT NECESSARILY a stationary point.**

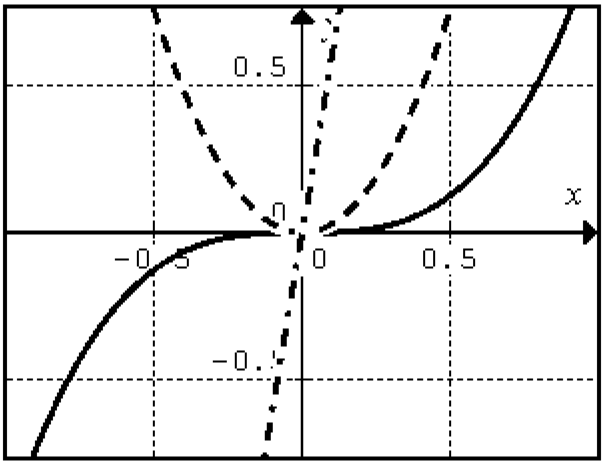


Figure 3 - Second derivative changes sign at point of inflection

- A point of inflection is a point at which the graph **changes concavity**.
- Formal definition:
  (x, y) is a stationary point of $f(x)$ if $f''(x) = 0 \wedge f''(x-\epsilon), f''(x+\epsilon)$ of different sign for any $\epsilon > 0$, OR if $f'(x-\epsilon), f'(x+\epsilon)$ are of similar sign.
- <u>**IMPORTANT**</u> – a point is **NOT NECESSARILY A POINT OF INFLECTION** if $d^2y/dx^2 = 0$ at the point. Additional information is required. Consider $y = x^4$.

Example

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^3 + 6x^2 - 12$. Find the points of inflection on $f(x)$.

$f'(x) = 6x^2 + 12x$, $f''(x) = 12x + 12$.

$f''(x) = 0 \Rightarrow x = -1$. $f''(x+\epsilon)$, $f''(x-\epsilon)$ of different sign.

$\therefore x = -1$ a point of inflection, $f(-1) = -2 + 6 - 12 = -8$

$\therefore (-1, -8)$ inflection

Find the points of inflection on $y = \dfrac{2x}{x^2+1}$, and sketch the graph

$$\frac{dy}{dx} = \frac{2(x^2+1) - (2x)(2x)}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}.$$

$$\frac{d^2y}{dx^2} = \frac{(-4x)(x^2+1)^2 - 2x(2)(x^2+1)}{(x^2+1)^4} = \frac{-4x^5-12x^3-8x}{(x^2+1)^4} = 0.$$

$$\therefore x(-4x^4-12x^2-8) = 0 \Rightarrow x = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0+0.01} = -0.11996, \quad \left.\frac{d^2y}{dx^2}\right|_{x=0-0.01} = 0.11996.$$

$\therefore \dfrac{d^2y}{dx^2}$ changes sign at $x=0$. $\therefore$ Inflection at $x=0$, $y=0$.

$\therefore$ Inflection at $(0,0)$.

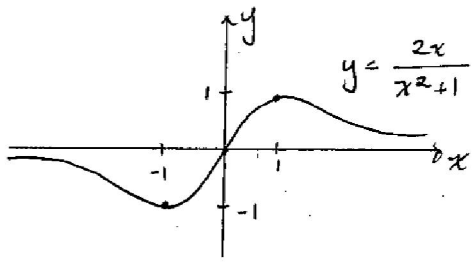
$$\frac{dy}{dx} = \frac{2-2x^2}{(x^2+1)^2} = 0 \Rightarrow 2-2x^2=0 \Rightarrow x = \pm 1.$$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=\pm 1} = \mp 1$. $\therefore x=1$, Local max, $x=-1$, local min.

$$y|_{x=1} = \frac{2}{2} = 1, \quad y|_{x=-1} = -\frac{2}{2} = -1.$$

$\therefore$ Max at $(1, 1)$, Min at $(-1, -1)$

$$\lim_{x \to \pm\infty} \frac{2x}{x^2+1} = 0.$$



Absolute Minima/Maxima

- M is the absolute maxima of a continuous function f on the interval [a, b] if $f(x) \leq M \ \forall \ x \in [a, b]$.
- N is the absolute minima of a continuous function f on the interval [a, b] if $f(x) \geq N \ \forall \ x \in [a, b]$.

Absolute minima/maxima may be stationary points OR endpoints. The only way to find out is to list all stationary points, and also find the endpoints (or limits as $x \to \pm\infty$) of the function.

Related Rates

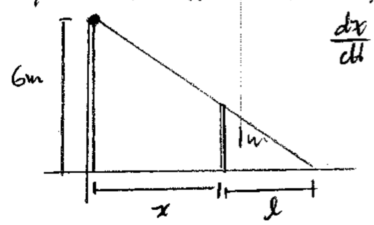
These questions often model a physical situation, e.g. water being poured into a cone, and give one rate, and ask for another. Solutions **always** use the chain rule.

Steps

- Write down all given information
- Use the chain rule to write an equation for the desired rate
- Differentiate a volume or area formula (usually) and substitute given values

Example

a street light hangs 6m high. a child of height 1m walks away from it at $1/4 \text{ m s}^{-1}$. Find the rate of change of shadow length.



$\frac{dx}{dt} = \frac{1}{4}$ $\quad \therefore \frac{dl}{dt} = \frac{dl}{dx} \cdot \frac{dx}{dt}$

$\frac{l+x}{6} = \frac{l}{1} \Rightarrow l + x = 6l \Rightarrow x = 5l \Rightarrow l = \frac{x}{5}$

$\therefore \frac{dl}{dx} = \frac{1}{5}$

$\therefore \frac{dl}{dt} = \frac{1}{5} \times \frac{1}{4} = \frac{1}{20} \text{ m s}^{-1}$

<u>Rational Functions and Reciprocal Functions</u>

A *rational function* is a function of the form $f(x) = P(x)/Q(x)$, where P and Q are polynomials.

A *reciprocal function* is a function of the form $f(x) = 1/g(x)$, where g is any function.

These graphs can be sketched using limits and differential calculus.

Rational Functions

Steps

- Find asymptotes
  - For vertical asymptotes, check for where the denominator is zero.
  - For horizontal asymptotes, compute $\lim\limits_{x \to \pm\infty} f(x)$, to see what the function approaches.
    - $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$ means that there is **no horizontal asymptote**

  - Other asymptotes (not necessarily straight line!) can be found from algebraic manipulation of the graph (for example, the function may be of the form $f(x) = \frac{1}{x} + g(x)$ so you will know that as $x \to \pm\infty, f(x) \to g(x)$, or that $y = g(x)$ is an asymptote.
- Find x/y intercepts
- Find turning points/stationary points, their nature must be stated.

For simple rational functions such as $y = x^2 + 1 + \frac{2}{x}$, addition of ordinates may be used instead to graph the function.

Reciprocal Functions
Sketching these graphs is somewhat less complicated than for rational functions.

Steps

- Find asymptotes
  - For vertical asymptotes, check for where the denominator is zero.
  - Horizontal asymptotes will exist where the function $f(x) \to \pm\infty$, since $\frac{1}{f(x)} \to 0$.
  -
- Find x/y intercepts
- Find turning points/stationary points, their nature must be stated.
  - In this case, differentiate $f(x)$, the denominator, and find its stationary points. From these, we may infer the nature of stationary points on $1/f(x)$ using the following rules:

Rules

- $f(x), 1/f(x)$ are always of the same sign
- $f(x)$ increasing $\Rightarrow 1/f(x)$ decreasing
- Local max of $f(x) \Rightarrow$ local min of $1/f(x)$
- Local min of $f(x) \Rightarrow$ local max of $1/f(x)$
- $y = f(x), y = 1/f(x)$ intersect wherever $y = \pm 1$.

Examples

Sketch $y = x - 5 + \frac{4}{x}$.

Asymptotes: As $x \to \infty$, $y \to x - 5$. $\therefore y = x - 5$.

V.A. at $x = 0$.

x-ints: $x - 5 + \frac{4}{x} = 0 \Rightarrow \frac{4}{x} = 5 - x \Rightarrow 4 = 5x - x^2$

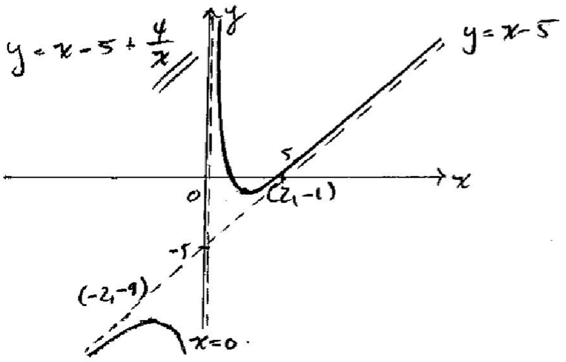
$\therefore x^2 - 5x + 4 = (x-1)(x-4) = 0 \quad \therefore x = 1, 4.$

y-ints: None

$\frac{dy}{dx} = 1 - \frac{4}{x^2} = 0 \Rightarrow 1 = \frac{4}{x^2} \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$

$\frac{d^2y}{dx^2} = \frac{8}{x^3}$. $\quad x = +2,\ \frac{d^2y}{dx^2} = 1 \quad \therefore$ Min at $x = 2$, $y = -1$. $\Rightarrow (2, -1)$

$x = -2,\ \frac{d^2y}{dx^2} = -1 \quad \therefore$ Max at $x = -2$, $y = -9$. $\Rightarrow (-2, -9)$



Sketch $y = \frac{1}{x^2 - 2x}$

$y = \frac{1}{x^2 - 2x} = \frac{1}{x(x-2)}$. Asymptotes: $x(x-2) = 0$

$\therefore x = 0,\ x = 2.$

$\lim_{x \to \infty} \frac{1}{x^2 - 2x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x^2 - 2x} = 0.$

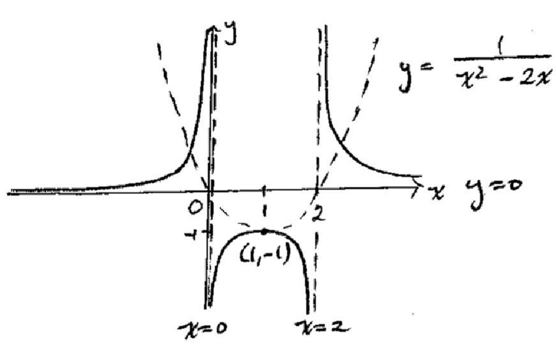
$\therefore y = 0.$

x-ints: none

y-ints: $y|_{x=0}$ = undefined.

$\frac{dy}{dx} = \frac{-(2x-2)}{(x^2-2x)^2} = \frac{2-2x}{(x^2-2x)^2} = 0 \Rightarrow x = 1.$

$y|_{x=1} = \frac{1}{1-2} = -1.$ $\therefore$ Stat. pt. on $y = \frac{1}{x^2 - 2x}$ at $(1, -1)$. (max).



Integral calculus

Integration Techniques

Antiderivatives of Standard Functions

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{1}{ax+b} \; dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int (ax+b)^n \; dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C, n \neq -1$$

$$\int e^{ax+b} \; dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \operatorname{cosec}^2(x) \; dx = -\cot(x) + C$$

$$\int \sec^2(x) \; dx = \tan(x) + C$$

$$\int \sec(x)\tan(x) \, dx = \sec(x) + C$$

$$\int \operatorname{cosec}(x)\cot(x) \, dx = -\operatorname{cosec}(x) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \; dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2 - x^2}} \; dx = \arccos(x) + C$$

$$\int \frac{a}{a^2 + x^2} \; dx = \arctan\left(\frac{x}{a}\right) + C$$

Integration Rules

Integration by Substitution

$$\int f(x) \; dx = \int f(x)\frac{dx}{du} \; du$$

Substitution Rule for Definite Integration

$$\int_a^b f(x)dx = \int_{u(a)}^{u(b)} f(x)\frac{dx}{du} \; du$$

Integration by Parts **(Not in course)**[1]

$$\int uv' \; dx = uv - \int vu' \; dx$$

Derivation

$$\frac{d}{dx}(uv) = u'v + uv'$$

$$uv' = \frac{d}{dx}(uv) - u'v$$

$$\int uv' dx = uv - \int vu' dx$$

Useful Trigonometric Identities

[1] Integration by parts can be used, however, but the formula must be derived. The derivation is included.

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Integration by Substitution

Using the substitution rule, a range of functions may be integrated.

Example

Find $\int \frac{3x^2}{x^3+5}\,dx$. let $u = x^3+5$ $\therefore \frac{dx}{du} = \frac{1}{3x^2}$

$\int \frac{3x^2}{x^3+5}\,dx = \int \frac{\cancel{3x^2}}{u} \cdot \frac{1}{\cancel{3x^2}}\,du = \int \frac{1}{u}\,du = \ln|u| + c$

$= \ln|x^3+5| + c$.

Find $\int \sin^3 x\,dx$.

$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$, $u = \cos x$ $\therefore \frac{dx}{du} = -\frac{1}{\sin x}$

$\therefore \int (1-\cos^2 x)\sin x\,dx = \int (1-u^2)\cancel{\sin x} \cdot \frac{-1}{\cancel{\sin x}}\,du$

$= \int (u^2-1)\,du = \frac{u^3}{3} - u + c = \frac{\cos^3 x}{3} - \cos x + c$.

Integration by Parts

Although not in the study design for SM, "integration by recognition" is in fact a special case of this. This method may be used in the SM examination, but the integration by parts rule must be derived before use.

Example

Find (a) $\int x\cos x\,dx$. (b) $\int \ln(2+3x)\,dx$ (c) $\int \sin^{-1}(x)\,dx$

(a) $\int x\cos x\,dx = \int x(\sin x)'\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

(b) $\int \ln(2+3x)\,dx = \int (x)'\ln(2+3x)\,dx = x\ln(2+3x) - \int x\cdot\frac{3}{2+3x}\,dx$

$= x\ln(2+3x) - \int \frac{3x}{2+3x}\,dx = x\ln(2+3x) - \int \frac{3x+2-2}{2+3x}\,dx$

$= x\ln(2+3x) - \int 1 - \frac{2}{2+3x}\,dx$

$= x\ln(2+3x) - \int dx + 2\int \frac{1}{2+3x}\,dx$

$= x\ln(2+3x) - x + \frac{2}{3}\ln|2+3x| + c$

(c) $\int \sin^{-1}(x)\,dx = \int (x)'\sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$\int \frac{x}{\sqrt{1-x^2}}dx$, let $u = 1-x^2$ $\therefore \frac{dx}{du} = -\frac{1}{2x}$

$\therefore \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{x}{\sqrt{u}}\cdot -\frac{1}{2x}\,du = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}\cdot\frac{\sqrt{u}}{1/2} + c = -\sqrt{u} + c$

$= -\sqrt{1-x^2} + c$

$\therefore \int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + c$

Partial Fractions

A useful technique to integrate some expressions.

Example

Find $\int \frac{1}{x^2+3x+2}\,dx$

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{(x+2)(x+1)}\,dx$$

$$\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1} \quad \therefore 1 = A(x+1) + B(x+2)$$

$$\therefore 1 = -A \Rightarrow A = -1. \qquad \therefore \frac{1}{(x+2)(x+1)} = \frac{-1}{x+2} + \frac{1}{x+1}$$

$$1 = B \Rightarrow B = 1.$$

$$\therefore \int \frac{1}{x^2+3x+2}\,dx = \int \left[\frac{-1}{x+2} + \frac{1}{x+1}\right] dx = -\ln|x+2| + \ln|x+1| + c$$

$$= \ln\left|\frac{x+1}{x+2}\right| + c$$

Find $\int \frac{x^2+3x+4}{(x+1)(x^2+1)}\,dx$.

$$\frac{x^2+3x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \therefore x^2+3x+4 = A(x^2+1) + (Bx+C)(x+1).$$

$$x = -1: \quad 1 - \cancel{3+4} = 2A \Rightarrow 2A = 2 \Rightarrow A = 1.$$

$$\therefore \cancel{x^2} + 3x + 4 = \cancel{x^2} + 1 + Bx^2 + Bx + Cx + C$$

$$\therefore 3x + 3 = Bx^2 + (B+C)x + C$$

$$\therefore C = 3, \quad 3x+3 = Bx^2 + (B+3)x + 3 \Rightarrow B = 0.$$

$$\therefore A = 1, \quad B = 0, \quad C = 3$$

$$\therefore \frac{x^2+3x+4}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{3}{x^2+1}$$

$$\therefore \int \frac{x^2+3x+4}{(x+1)(x^2+1)}\,dx = \int \frac{1}{x+1} + \frac{3}{x^2+1}\,dx = \ln|x+1| + 3\tan^{-1}(x) + c$$

Integration using Trigonometric Identities

Identities may be used to integrate expressions of the form $\sin^m(x)\cos^n(x)$. There are 3 cases to be considered.

1. ***m* odd, *n* even.**
   Thus, $m = 2k+1$, $\sin^m(x)\cos^n(x) = \sin^{2k}(x)\cdot\sin(x)\cos^n(x)$
   $$= (\sin^2 x)^k \cdot \sin(x)\cos^n(x)$$
   $$= (1-\cos^2 x)^k \cdot \cos^n(x)\cdot\sin(x)$$
   This expression can now be integrated by substituting $u = \cos(x)$.
2. ***m* even, *n* odd**
   Thus $n = 2k+1$, $\sin^m(x)\cos^n(x) = \sin^m(x)\cos^{2k}(x)\cdot\cos(x)$
   $$= \sin^m(x)\cdot(1-\sin^2 x)^k\cdot\cos(x)$$
   This expression can now be integrated by substituting $u = \sin(x)$

3. **Both m, n even**
   Use $\sin(2x) = 2\sin(x)\cos(x)$ or $\sin^2 x = \frac{1-\cos(2x)}{2}, \cos^2 x = \frac{1+\cos(2x)}{2}$

Other trigonometric identities may also be used for integration. For example, the integral of $\tan^2 x$ may be found by using the identity $1 + \tan^2(x) = \sec^2(x)$.

Integration by Partial Fractions

The rules for partial fraction are as follows:

- For every linear term $(ax + b)$ in the denominator, there is a partial fraction $\frac{A}{ax+b}$, where A is some constant
- For a repeated linear term $(ax + b)^n$ in the denominator, there are n partial fractions $\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \cdots, \frac{A_n}{(ax+b)^n}$.
- For a quadratic term $ax^2 + bx + c$ in the denominator, there is a partial fraction $\frac{Ax+B}{ax^2+bx+c}$.
- The rules for partial fractions **only apply** where the degree of the numerator polynomial is less than that of the denominator polynomial. If this is not the case, polynomial long division must first be applied to obtain a fraction that can be expanded.

Example:

Find $\int \frac{1}{x^2+3x+2}\,dx$

$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{(x+2)(x+1)}\,dx$

$\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$ $\therefore 1 = A(x+1) + B(x+2)$

$\therefore 1 = -A \Rightarrow A = -1.$
$1 = B \Rightarrow B = 1.$
$\therefore \frac{1}{(x+2)(x+1)} = \frac{-1}{x+2} + \frac{1}{x+1}$

$\therefore \int \frac{1}{x^2+3x+2}\,dx = \int \left[\frac{-1}{x+2} + \frac{1}{x+1}\right] dx = -\ln|x+2| + \ln|x+1| + c$
$= \ln\left|\frac{x+1}{x+2}\right| + c$

Find $\int \frac{x^2+3x+4}{(x+1)(x^2+1)}\,dx$.

$\frac{x^2+3x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$. $\therefore x^2+3x+4 = A(x^2+1)+(Bx+C)(x+1)$.

$x=-1$: $1-\cancel{3}^{1}\cancel{+4} = 2A \Rightarrow 2A = 2 \Rightarrow A=1$.

$\therefore \cancel{x^2}+3x+4 = \cancel{x^2}+1 + Bx^2+Bx+Cx+C$

$\therefore 3x+3 = Bx^2+(B+C)x+C$

$\therefore C=3, \quad 3x+3 = Bx^2+(B+3)x+3 \Rightarrow B=0$.

$\therefore A=1, \quad B=0, \quad C=3$

$\therefore \frac{x^2+3x+4}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{3}{x^2+1}$

$\therefore \int \frac{x^2+3x+4}{(x+1)(x^2+1)}\,dx = \int \frac{1}{x+1} + \frac{3}{x^2+1}\,dx = \ln|x+1| + 3\tan^{-1}(x)+c$

Integration by Recognition

This technique is "invented" by VCAA to compensate for the non-inclusion of integration by parts from the course. It is included for completeness.

In order to integrate some function $f(x)$, first differentiate $xf(x)$ then solve for $f(x)$

$$\frac{d}{dx}[xf(x)] = f(x) + xf'(x)$$
$$f(x) = \frac{d}{dx}[xf(x)] - xf'(x)$$
$$\int f(x)dx = xf(x) - \int xf'(x)dx$$

This is useful to integrate functions such as $\ln(x)$, $\arcsin(x)$.

Example

Find $\int [\ln(x)]^2 dx$.

$\frac{d}{dx}\left[x(\ln x)^2\right] = [\ln(x)]^2 + \frac{1}{x}\,2\ln x \cdot x = (\ln x)^2 + 2\ln x$

$\therefore (\ln x)^2 = \frac{d}{dx}[x(\ln x)^2] - 2\ln x$

$\therefore \int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x\, dx$

$\frac{d}{dx}x\ln x = \ln x + \frac{1}{x}\cdot x = \ln x + 1 \Rightarrow \ln x = \frac{d}{dx}x\ln x - 1$

$\int \ln x\, dx = x\ln x - x + c$

$\therefore \int [\ln(x)]^2 dx = x(\ln x)^2 - 2x\ln x + 2x + c$

The Definite Integral

Definition

The definite integral is defined as the limit of the **Riemann Sum**, i.e. limit as $n \to \infty$ of n rectangles under the curve $f(x)$ in the interval $[a, b]$, approximating the area under the curve, that is:

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

Fundamental Theorem of Calculus

$$\boxed{\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)}$$

Where F is **any antiderivative** of f.

Useful Properties of the Definite Integral

- $\int_a^b f(x)dx = \int_a^b f(u)du$, that is, they are **numerically equal.**
- If $f(x)$ is odd, then $\int_{-a}^{a} f(x)dx = 0$
- If $f(x)$ is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

It is expected that the other, easier to remember properties have been retained from previous studies (MM).

Examples

E50. Evaluate: (a) $\int_0^{\pi/2} e^{\cos x} \sin x \, dx$ (b) $\int_0^{\pi/4} \tan^6 x \sec^2 x \, dx$

(c) $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$ (d) $\int_0^{\frac{1}{2}} \frac{dx}{x+2}$. (e) $\int_3^4 \frac{dx}{25-x^2}$

(a) $\int_0^{\frac{\pi}{2}} e^{\cos x} \sin x \, dx$, let $u = \cos x$ $\therefore \frac{dx}{du} = \frac{-1}{\sin x}$. $u|_{x=0} = 1$, $u|_{x=\frac{\pi}{2}} = 0$.

$\therefore \int_0^{\frac{\pi}{2}} e^{\cos x} \sin x \, dx = \int_1^0 e^u \cancel{\sin x} \cdot \frac{-1}{\cancel{\sin x}} du = \int_0^1 e^u \, du = \left[e^u\right]_0^1 = e^1 - e^0$

$= e - 1.$

(b) $\int_0^{\pi/4} \tan^6 x \sec^2 x \, dx$ let $u = \tan x$ $\therefore \frac{dx}{du} = \frac{1}{\sec^2 x}$

$u|_{x=0} = 0$, $u|_{x=\frac{\pi}{4}} = 1$.

$\int_0^{\pi/4} \tan^6 x \sec^2 x \, dx = \int_0^1 u^6 \cancel{\sec^2 x} \cdot \frac{1}{\cancel{\sec^2 x}} du = \int_0^1 u^6 \, du = \left[\frac{u^7}{7}\right]_0^1 = \frac{1}{7}.$

(c) $\int_0^{\pi/2} \cos^3 x \, dx = \int_0^{\pi/2} (1 - \sin^2 x) \cos x \, dx$. let $u = \sin x$ $\therefore \frac{dx}{du} = \frac{1}{\cos x}$

$u|_{x=0} = 0$, $u|_{x=\frac{\pi}{2}} = 1$.

$\int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \cos x \, dx = \int_0^1 (1 - u^2) \cancel{\cos x} \frac{1}{\cancel{\cos x}} du = \int_0^1 (1 - u^2) du$

$= \int_1^0 (u^2 - 1) du = \left[\frac{u^3}{3} - u\right]_1^0 = -\left(\frac{1}{3} - 1\right) = +\frac{2}{3}$

(d) $\int_0^{\frac{1}{2}} \frac{dx}{x+2} = \left[\ln|x+2|\right]_0^{1/2} = \ln\left|\frac{1}{2} + 2\right| - \ln 2 = \ln\left|\frac{5}{2}\right| - \ln 2 = \ln 5 - \ln 2 - \ln 2$

$= \ln\left|\frac{5}{4}\right| = \ln \frac{5}{4}$

(e) $\int_3^4 \frac{dx}{25-x^2}$. $\frac{1}{25-x^2} = \frac{1}{(5-x)(5+x)} = \frac{A}{5-x} + \frac{B}{5+x}$

$\therefore 1 = A(5+x) + B(5-x)$

$\therefore 1 = 10A \Rightarrow A = \frac{1}{10}$

$1 = 10B \Rightarrow B = \frac{1}{10}$.

$\therefore \int_3^4 \frac{dx}{25-x^2} = \int_3^4 \left[\frac{1}{10(5-x)} + \frac{1}{10(5+x)}\right] dx$

$= \left[\frac{-1}{10} \ln|5-x| + \frac{1}{10} \ln|5+x|\right]_3^4$

$= \left[\frac{1}{10} \ln\left|\frac{5+x}{5-x}\right|\right]_3^4$

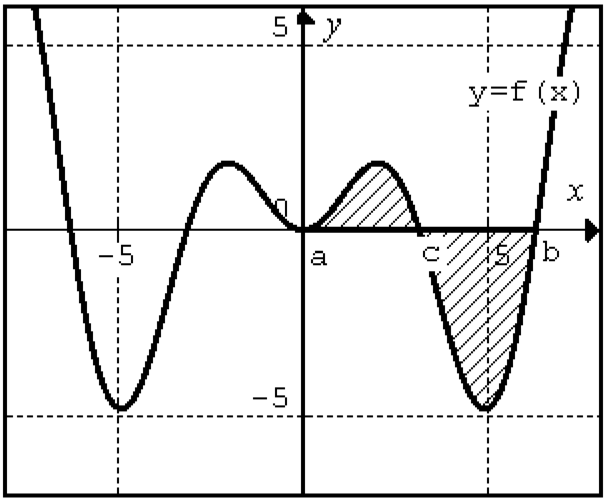
$= \frac{1}{10} \ln\left|\frac{9}{1}\right| - \frac{1}{10} \ln\left|\frac{8}{2}\right|$

$= \frac{1}{10} \ln\left|\frac{9}{4}\right| = \frac{1}{5} \ln\left(\frac{3}{2}\right)$

Signed Area

- Regions above the x-axis have positive signed area
- Regions below the x-axis have negative signed area.

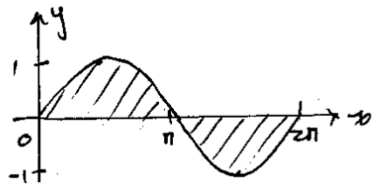
When asking for the 'total area', the question means the area regardless of sign. Total area can be computed by negating the negative parts of the area.



$$\int_a^b f(x)\,dx = \int_a^c f(x)dx - \int_c^b f(x)\,dx$$

Example:

Find the total area between $y=\sin(x)$ and the x-axis between $x=0$ and $x=2\pi$.



$$A = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

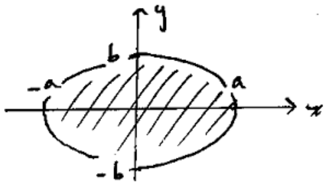
$$= 2\int_0^{\pi} \sin x\,dx \quad \text{(By property of } \sin x\text{)}$$

$$= 2\left[-\cos x\right]_0^{\pi} = 2\left[\cos x\right]_{\pi}^{0}$$

$$= 2(\cos 0 - \cos \pi) = 2(1+1) = 4. \text{ units}^2$$

Show that the area of an ellipse is πab.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$



$A = 4\int_0^a y\,dx.$ $\quad \frac{y^2}{b^2} = 1-\frac{x^2}{a^2} \Rightarrow y^2 = b^2-\frac{b^2}{a^2}x^2$

$y = \pm b\sqrt{1-\frac{x^2}{a^2}}.$

In the first quadrant, $y = b\sqrt{1-\frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}$

$\therefore A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx.$ Let $x = a\sin\theta$. $(x\in[-a,a])$ $(\theta\in[-\frac{\pi}{2},\frac{\pi}{2}])$

$\therefore \frac{dx}{d\theta} = a\cos\theta$, $\theta|_{x=0} = 0$, $\theta|_{x=a} = \frac{\pi}{2}$

$$\therefore 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = 4\int_0^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2-a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{b}{\cancel{a}}\cdot \cancel{a}\sqrt{1-\sin^2\theta}\cdot a\cos\theta\, d\theta = 4ab\int_0^{\pi/2}\cos\theta\cdot\cos\theta\, d\theta$$

$$= 4ab\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta = 4ab\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}\,d\theta = 2ab\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

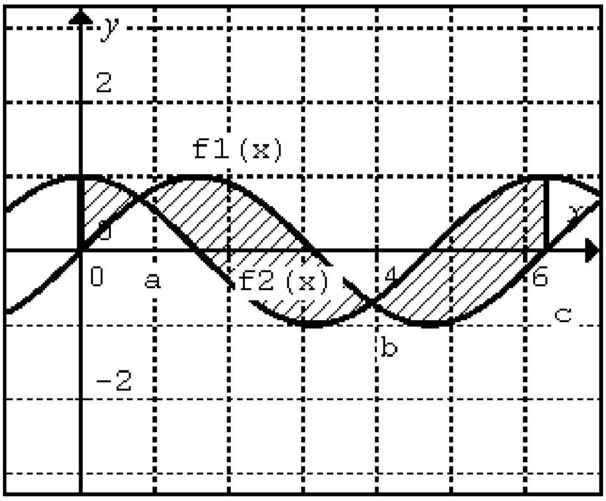
$$= 2ab\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left[\frac{\pi}{2}+\cancel{\frac{1}{2}\sin\pi}\right] = \pi ab \text{ units}^2$$

Applications of Integration

Bounded Area – the area bounded by two curves can be found by using the following formula:

$$A = \int_a^b (\text{upper curve}) - (\text{bottom curve})dx$$

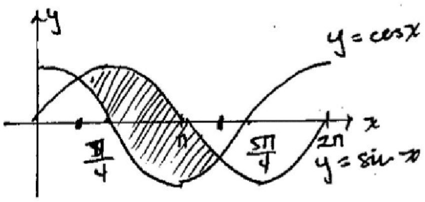
If the graphs cross, a separate integral will have to be computed for each interval, that is:



$$A = \int_0^a f_2(x) - f_1(x)\,dx + \int_a^b f_1(x) - f_2(x)dx + \int_b^c f_2(x) - f_1(x)dx$$

Example

Find the bounded area between $y = \sin x$ and $y = \cos x$.



Intersection:

$\sin x = \cos x \quad \therefore \tan x = 1.$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$\therefore A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx = \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4} = \left[\sin x + \cos x\right]_{\frac{5\pi}{4}}^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}\right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 4 \cdot \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \text{ units}^2$$

Volumes of Solids of Revolution

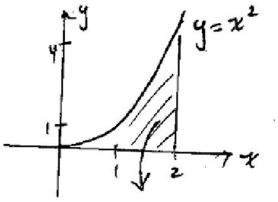
The volume of a solid of revolution can be computed using the formula:

$$V = \pi \int_a^b y^2 \, dx \text{ or } V = \int_a^b x^2 \; dy$$

The first formula is used for integrating along the x-axis, the second for the y-axis.

Example:

E54. Find the volume of the solid of revolution obtained by rotating $y = x^2$ (a) about the x-axis, between $x = 1$, $x = 2$
(b) about the y-axis, between $y = 3$, $y = 4$.



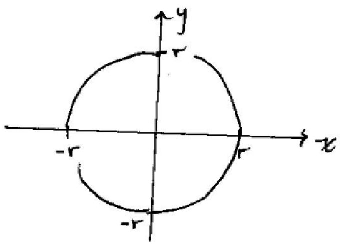
(a) $V = \pi \int_1^2 y^2 \, dx = \pi \int_1^2 x^4 \, dx = \pi \left[\frac{x^5}{5} \right]_1^2 =$

$= \pi \left(\frac{32}{5} - \frac{1}{5} \right) = \frac{31\pi}{5}$ units³

(b) $V = \pi \int_3^4 x^2 \, dy = \pi \int_3^4 y \, dy = \pi \left[\frac{y^2}{2} \right]_3^4$

$= \pi \left(\frac{16}{2} - \frac{9}{2} \right) = \pi \left(\frac{7}{2} \right) = \frac{7}{2}\pi$ units³

E55. Prove that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.



Rotate about x-axis.

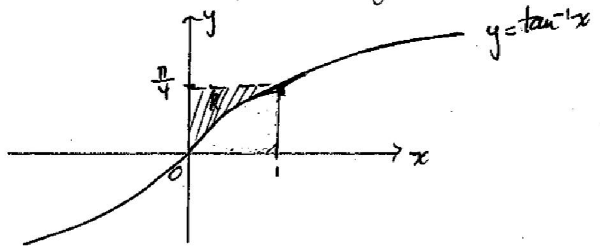
$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$V = \pi \int_{-r}^{r} y^2 \, dx = \pi \int_{-r}^{r} (r^2 - x^2) \, dx$

$= 2\pi \int_0^r (r^2 - x^2) \, dx = 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$

$= 2\pi \left(r^3 - \frac{r^3}{3} \right) = 2\pi \left(\frac{2r^3}{3} \right) = \frac{4\pi r^3}{3}$ units³

R is the region bounded by $y = \tan^{-1} x$, the y-axis, and $y = \frac{\pi}{4}$.
Find the volume formed by rotating R about the y-axis.



$V = \pi \int_0^{\frac{\pi}{4}} x^2 \, dy$. $\quad y = \tan^{-1} x \Rightarrow x = \tan y \Rightarrow x^2 = \tan^2 y$

$= \pi \int_0^{\pi/4} \tan^2 y \, dy = \pi \int_0^{\pi/4} (\sec^2 y - 1) \, dy = \pi \left[\tan y - y \right]_0^{\pi/4}$

$= \pi \left[1 - \frac{\pi}{4} \right] = \pi - \frac{\pi^2}{4}$ units^3

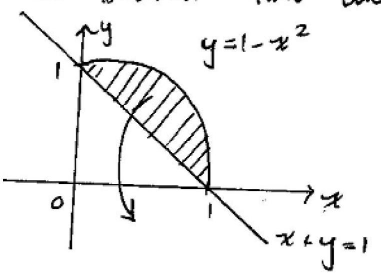
In the case that the region of rotation is bounded by two curves, the volume is given by:

$$V = \pi \int_a^b (y_1)^2 - (y_2)^2 \; dx \text{ or V} = \pi \int_a^b (x_1)^2 - (x_2)^2 dy$$

Where y_1, x_1 are the 'outer' radii, and y_2, x_2 are the inner radii.

Example

The region bounded by $y = 1 - x^2$ and $x + y = 1$ is rotated about the x-axis. Find the generated volume.



$V = \pi \int_0^1 (1 - x^2)^2 - (1 - x)^2 dx$

$= \pi \int_0^1 \left[1 - 2x^2 + x^4 - (1 - 2x + x^2) \right] dx$

$= \pi \int_0^1 \left[\cancel{1} - 2x^2 + x^4 - \cancel{1} + 2x - x^2 \right] dx$

$= \pi \int_0^1 \left[x^4 - 3x^2 + 2x \right] dx$

$= \pi \left[\frac{x^5}{5} - x^3 + x^2 \right]_0^1$

$= \pi \left(\frac{1}{5} \cancel{-1+1} \right) = \frac{\pi}{5}$ units^3

A more general formula for finding volumes which can be used is

$$V = \int_a^b A(x)\,dx$$

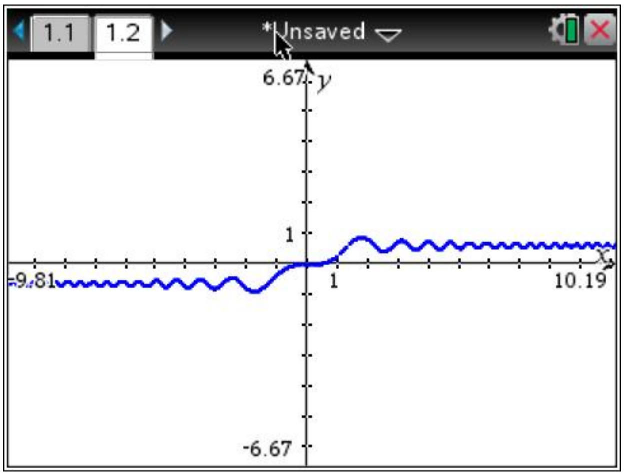
Where $A(x)$ is the cross-sectional area of a solid at a particular x-axis value.

Finding Values of the Antiderivative

The value of the antiderivative at any point can be computed if an initial value is known.

$$F(x) = \int_a^x f(t)\ \ dt + F(a)$$

This is useful when the antiderivative can only be approximated, allowing us to actually graph the function. For example, $\int \sin(x^2)\,dx$ can be graphed, assuming it passes through $(0,0)$:



Differential Equations

A differential equation is an equation that contains one or more derivatives. The solution of a differential equation is a relation which satisfies the differential equation.

The SM course covers a very limited range of differential equations. In this notes, we will first consider the SM-only types, and then we will move on to other types of ODEs.

Order - the order of a differential equation is the order of the highest derivative in the equation. For example, $y'' + 2y + 3 = 0$ is a second order ODE.

<u>**Specialist Mathematics Differential Equations**</u>

In SM, the study of ODEs is restricted to the following:

- $\frac{dy}{dx} = f(x)$
- $\frac{dy}{dx} = f(y)$
- $\frac{d^2y}{dx^2} = f(x)$
- $\frac{dy}{dx} = f(x)g(y)$ (a possibility on SACs, but not on the exam. Separable equations will be introduced in the 2016 study design)

Solution of these differential equations is fairly straightforward:

- $\frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)dx + C$
- $\frac{dy}{dx} = f(y) \Rightarrow \frac{dx}{dy} = \frac{1}{f(y)} \Rightarrow x = \int \frac{1}{f(y)}dy + C$
- $\frac{d^2y}{dx^2} = f(x) \Rightarrow y = \int [\int f(x)dx]\, dx + Cx + D$
- $\frac{dy}{dx} = f(x)g(y)$, see the section on **separable equations**.

<u>**Applications of Differential Equations**</u>

Differential equations occur in a range of physical phenomena, for example, radioactive decay, chemical reaction rates, etc.

Inflow-Outflow Problems

These problems are quite popular in SM, involving a substance flowing into a container at a certain rate and a thoroughly mixed mixture being drawn out usually at the same rate. A general method of solution can be used.

$$\boxed{\frac{dQ}{dt} = [\text{inflow rate}] \cdot [\text{concentration}] - [\text{outflow rate}] \cdot \frac{\mathrm{Q}}{\mathrm{V(t)}}}$$

Where Q is the amount of substance at time t, and $V(t)$ is the volume at time t. Commonly, V is a constant if [inflow rate]=[outflow rate].

> Example
> A tank holds 50 L of water in which 20g of HCl is dissolved. HCl of concentration 0.1g/L is being added at the rate of 2 L/min, and the solution is constantly mixed and withdrawn at he same rate. Find Q, the amount of HCl (in g) at time t. What is the limiting value of Q?

A general solution for this type of question can be found:

Suppose inflow rate/outflow rate R with inflow concentration C, and volume V. Let Q be the quantity of substance.

$$\frac{dQ}{dt} = CR - R \cdot \frac{Q}{V} = \frac{CRV - RQ}{V}$$

$$\frac{dt}{dQ} = \frac{V}{CRV - RQ} = \frac{1}{R} \cdot \frac{V}{CV - Q}$$

$$t = \frac{1}{R} \int \frac{V}{CV - Q} \, dQ = \frac{1}{R} \cdot V \cdot (-1) \ln|CV - Q| + c$$

$$t = -\frac{V}{R} \ln|CV - Q| + c$$

$$\ln|CV - Q| = -\frac{Rt}{V} + c$$

$$CV - Q = \pm e^{-\frac{Rt}{V} + c} = Ae^{-\frac{Rt}{V}}$$

$$Q = CV - Ae^{-\frac{Rt}{V}}$$

Where A is an arbitrary constant to be found from initial conditions.

Chain Rule
The chain rule can also be used to set up a differential equation for a physical situation. For example, if the rate of change of height is known, dh/dt, one can set up an equation for volume:

$$\frac{dv}{dt} = \frac{dv}{dh} \frac{dh}{dt}$$

Numerical Methods
The solution to a differential equation can be approximated by a range of numerical methods.

<u>Definite Integral</u>

The solution to a differential equation can be found by the use of a definite integral. From FTC:

$$f(x) = \int_a^x f'(t) dt + f(a)$$

Thus specific values of a solution curve can be found if a certain coordinate is known.

For example, $\frac{dy}{dx} = e^{x^2}$ can be found at x=1 given it passes through (0, 0):

$$y(0) = \int_0^1 e^{x^2} dx \approx 1.463$$

Euler's Method

Euler's Method is a method of solution of first order equations, based on linear approximation. Given a step size h:

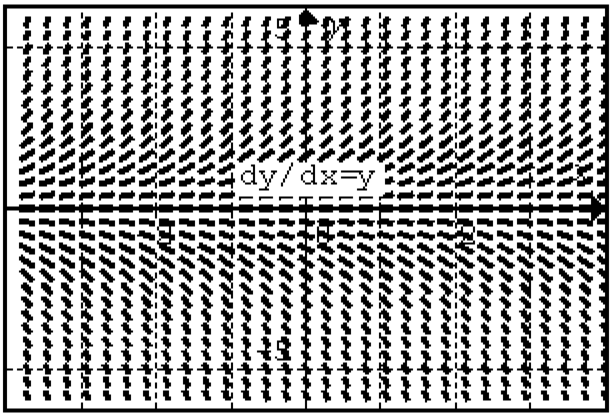
$$y_{i+1} = y_i + hy'(x_{i+1}), \quad x_{i+1} = x_i + h$$

Where y_0 is the value of y at x_0.

Example: If $\frac{dy}{dx} = \frac{x^2}{y}$ and $(x_0, y_0) = (1, 1)$, find $y(1.2)$ using $h = 0.1$
CAS Calculator Syntax: *##insert cas calculator syntax##*

Direction Fields

A direction field for a differential equation is a method of visualisation in which the value of dy/dx is computed for points (x, y), and a small line segment plotted at that point with the computed slope. For example, the direction field for the equation $\frac{dy}{dx} = y$ looks like:



In order to plot differential equations by hand, one must find dy/dx for many coordinates (x, y) within the region of interest. In exams, a calculator can usually be used to assist with this.

Another method of plotting differential equations is to plot them as "equigradient" lines, i.e. curves along which dy/dx has the same value.

This can be found by letting $dy/dx = c$, where c is the required slope.

IMPORTANT: The following information is not relevant to the VCAA SM Study Design.

First Order Differential Equations

First order differential equations can be classified as **separable, homogeneous, or linear.**

Separable Equations

A separable equation is of the form

$$\frac{dy}{dx} = f(x)g(y)$$

The solution to this differential equation can be obtained by direct integration:

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{1}{g(y)}dy = f(x)dx$$

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

Example

Solve $\frac{dy}{dx} = \frac{x^2}{y}$

$$\int y\ dy = \int x^2\ dx \Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C \Rightarrow 3y^2 = 2x^3 + C$$

Solve the following differential equations:

(a) $\frac{dy}{dx} = \frac{y}{x^2-1}$ (b) $\frac{dy}{dx} = \frac{x^2+1}{xy(y+1)}$ (c) $\cos^2 x \frac{dy}{dx} = 1 - y^2$.

(a) $\frac{dy}{dx} = \frac{y}{x^2-1}$ $\therefore \int \frac{1}{y}\,dy = \int \frac{1}{x^2-1}\,dx$

$\int \frac{1}{x^2-1}\,dx = -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|$ $\therefore \ln|y| = -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + c$.

$\therefore y = e^{-\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + c} = Ce^{-\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|} = Ce^{\ln\sqrt{\frac{x-1}{x+1}}} = C\sqrt{\frac{x-1}{x+1}}$.

(b) $\frac{dy}{dx} = \frac{x^2+1}{xy(y+1)}$ $\therefore \int \frac{1}{y(y+1)}\,dy = \int \frac{x^2+1}{x}\,dx = \int x + \frac{1}{x}\,dx$

LHS: $\int \frac{1}{y(y+1)}\,dy$: $\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow 1 = A(y+1) + By$.

$\therefore 1 = A$, $1 = -B \Rightarrow B = -1$.

$\therefore \int \frac{1}{y(y+1)}\,dy = \int \frac{1}{y} - \frac{1}{y+1}\,dy = \ln|y| - \ln|y+1| + c = \ln\left|\frac{y}{y+1}\right| + c$.

$\therefore \ln\left|\frac{y}{y+1}\right| = \int\left(x + \frac{1}{x}\right)dx = \frac{x^2}{2} + \ln|x| + c$

$\therefore \ln\left|\frac{y}{y+1}\right| = \frac{x^2}{2} + \ln|x| + c$.

(c) $\cos^2 x \frac{dy}{dx} = 1 - y^2$ $\therefore \frac{dy}{dx} = \frac{1-y^2}{\cos^2 x} = (1-y^2)\sec^2 x$

$\therefore \int \frac{1}{1-y^2}\,dy = \int \sec^2 x\,dx$ $\therefore$ LHS: $\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}$

$\therefore 1 = A(1+y) + B(1-y)$

$\therefore 1 = 2A \Rightarrow A = \frac{1}{2}$.

$1 = 2B \Rightarrow B = \frac{1}{2}$

$\therefore \int \frac{1}{1-y^2}\,dy = \int\left(\frac{1}{2(1-y)} + \frac{1}{2(1+y)}\right)dy$

$= -\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| + c = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| + c$

$\therefore \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \tan x + c$.

Homogeneous Equations

A differential equation expressible in the form $\frac{dy}{dx} = f(\frac{y}{x})$ can be reduced to a separable equation by the introduction of a third variable, $v = y/x$.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\therefore x\frac{dv}{dx} + v = f(v) \Rightarrow \frac{dv}{dx} = \frac{f(v) - v}{x}$$

I.e. we have obtained a separable equation which can now be solved for v and hence for y.

Example:

Solve: $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$. This equation is a first-order homogeneous equation.

Let $y = vx$ $\quad\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x \cdot vx} = \frac{x^2(1+v^2)}{2x^2 v}$

$\therefore x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2vx^2} - v = \frac{x^2 + x^2v^2 - 2v^2x^2}{2vx^2} = \frac{x^2 - x^2v^2}{2vx^2} = \frac{1-v^2}{2v}$

$\therefore \frac{dv}{dx} = \frac{1-v^2}{2v} \cdot \frac{1}{x} \Rightarrow \int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$

LHS: $\int \frac{2v}{1-v^2}\,dv$, Let $u = 1-v^2$ $\quad\therefore \frac{dv}{du} = \frac{-1}{2v}$

$\therefore \int \frac{2v}{1-v^2}\,dv = \int \frac{\cancel{2v}}{u} \cdot \frac{-1}{\cancel{2v}}\,du = -\ln|u| + c = -\ln|1-v^2| + c.$

$\therefore -\ln|1-v^2| = \ln|x| + c.$ Substitute $v = \frac{y}{x}$

$\therefore -\ln\left|1 - \frac{y^2}{x^2}\right| = \ln|x| + c$

Other Substitutions

Other substitutions can be made to transform an equation into one which can be solved. For example, $\frac{dy}{dx} = (4x - y + 1)^2$ cannot be solved using any of the above methods. However, if we substitute $v = 4x - y$:

$$\frac{dv}{dx} = 4 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 4 - \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} = 4 - (v+1)^2 \Rightarrow \int \frac{1}{4-(v+1)^2}\,dv = \int dx$$

We have obtained an equation which can be solved using conventional techniques.

Linear Equations

Linear equations are differential equations which can be expressed in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

A general solution may be used for the solution of these equations:

$$\boxed{y = \frac{\int \mu(x)Q(x)dx + C}{\mu(x)}, \mu(x) = e^{\int P(x)dx}}$$

Derivation
Suppose we have some function $\mu(x)$, so

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = Q(x)\mu(x)$$

Suppose that $\mu(x)P(x) = \mu'(x)$

$$\mu(x)\frac{dy}{dx} + \mu'(x)y = Q(x)\mu(x)$$

Applying the product rule:

$$\frac{d}{dx}(\mu(x) \cdot y) = Q(x)\mu(x)$$
$$\therefore \mu(x)y = \int Q(x)\mu(x)dx \Rightarrow y = \frac{\int Q(x)\mu(x)dx}{\mu(x)}$$

Finding $\mu(x)$:

$$\mu P = \frac{d\mu}{dx} \Rightarrow \int \frac{1}{\mu} d\mu = \int P(x)\ dx$$
$$\ln(\mu) = \int P(x)dx \Rightarrow \mu = e^{\int P(x)dx}$$

Second Order Differential Equations

Linear Equations – linear equations take the form $a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = f(x)$.

Homogeneous Equations
If $f(x) = 0$, the equation is **homogeneous**. Once again, these equations have a general solution.

Suppose some equation $y = e^{kx}$ satisfies the equation. Thus $\frac{dy}{dx} = ke^{kx}, \frac{d^2y}{dx^2} = k^2e^{kx}$

Thus

$$a_1k^2e^{kx} + a_2ke^{kx} + a_3e^{kx} = 0$$
$$e^{kx}(a_1k^2 + a_2k + a_3) = 0 \Rightarrow a_1k^2 + a_2k + a_3 = 0\ (\because e^{kx} \neq 0)$$

Clearly, if there is to be a solution to the equation, $a_1k^2 + a_2k + a_3 = 0$ must be satisfied. This equation is termed the **auxiliary equation** or **characteristic equation** of the differential equation, denoted $F(k) = a_1k^2 + a_2k + a_3$.

Definition

The **auxiliary equation** of a linear homogeneous equation $a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = 0$ is

$$F(k) = a_1 k^2 + a_2 k + a_3$$

$F(k)$ is a quadratic, and thus has two solutions, k_1, k_2. As a result, both $y = e^{k_1 x}$ and $y = e^{k_2 x}$ are solutions. A general solution can be developed from this as follows.

If k_1, k_2 are both real and distinct, the general solution is given by

$$\boxed{y = Ae^{k_1 x} + Be^{k_2 x}}$$

If k_1, k_2 are real and equal, the general solution is given by

$$\boxed{y = (Ax + B)e^{kx}}$$

If k_1, k_2 are complex, i.e. $k_1, k_2 = a \pm bi$, the general solution is given by

$$\boxed{y = e^{ax}(\alpha \cos(bx) + \beta \sin(bx)) \text{ or } y = Ae^{ax}(\sin(B + bx))}$$

Solution of Second Order Equations for Specific Solutions

At least two values of the function and its derivatives must be given in order to solve for a specific solution.

- Initial value problem (IVP) – $y(0), y'(0)$ are given
- Boundary value problem - $y(a), y(b)$ are given

Inhomogenous Equations

Inhomogeneous equations are of the form $a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = f(x)$, where $f(x) \neq 0$. There exists a relationship between the inhomogeneous equation and the corresponding homogeneous equation (**the complementary equation)** (with $f(x) = 0$).

Suppose we have a particular solution to the inhomogeneous equation, $y = u(x)$. Thus

$$a_1 u'' + a_2 u' + a_3 u = f(x) \ldots (1)$$

Again, suppose we have a general solution to the homogeneous equation, $y = v(x)$. Thus

$$a_1 v'' + a_2 v' + a_3 v = 0 \ldots (2)$$

Adding (1) and (2) gives

$$a_1 (u + v)'' + a_2 (u + v)' + a_3 (u + v) = f(x)$$

I.e. $y = u + v$ is a solution.

In general:

For any inhomogeneous linear equation, the solution is given by

$$\boxed{y = y_p + y_c}$$

Where y_p is a particular integral of the equation, and y_c is the **complementary function**, i.e. the general solution of the homogeneous equation.

Methods of Finding Particular Integrals

Method of Undetermined Coefficients

- **Polynomial function, where $\boldsymbol{f(x) = P_n(x)}$ a polynomial of degree n**
  The particular solution will have the same degree as $f(x)$ if $a_3 \neq 0$, $n+1$ degree if $a_3 = 0$, and $n+2$ degree if $a_2 = a_3 = 0$.
  Letting $y_p = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, we can solve for $a_n \dots a_0$ by equating coefficients.
- **Exponential Function, where $\boldsymbol{f(x) = Ce^{kx}}$.**
  Substitution of $y_p = Ae^{kx}$, and solve for A.
- **Trigonometric Functions**
  If $f(x) = K\cos(mx)$ or $K\sin(mx)$, let $y_p = a\cos(mx) + b\sin(mx)$. Solve for a, b.

Principle of Superposition

If there are multiple terms in the RHS, a separate DE is solved for each term. The solution is the sum of each of the individual solutions. This is because of the property of the derivative, $\frac{d}{dx}(u) + \frac{d}{dx}(v) = \frac{d}{dx}(u+v)$

Kinematics

Kinematics is the study of motion without reference to the causes of the motion. The topic here is largely just an application of topics from earlier on in the course.

- **Displacement** is denoted s or x
- **Velocity** is the rate of change of **displacement**, and is denoted v or $\dot{x}$
- **Acceleration** is the rate of change of **velocity**, and is denoted a or $\ddot{x}$

Examples

For a particle with displacement $x = -t^3 + 12t^2 - 6t + 4, t \geq 0$, determine when the acceleration is zero, and find the velocity at this instant. Is this velocity a maximum or minimum?

$x = -t^3 + 12t^2 - 6t + 4, \; t \geq 0.$

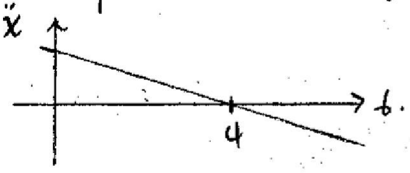
$\therefore \dot{x} = -3t^2 + 24t - 6 \Rightarrow \ddot{x} = -6t + 24.$

$\ddot{x} = -6t + 24 = 0 \Rightarrow t = \frac{-24}{-6} = 4 \text{ (s)}.$

$\therefore$ acceleration is zero at $t = 4$.

$\dot{x}(4) = -3(16) + 24(4) - 6 = 42 \text{ (m/s)}.$

$\therefore$ Speed is 42 m/s.



t		4	
$\ddot{x}$	+	0	−
$\dot{x}$	↗	—	↘

$\therefore t = 4, \; \dot{x} = 42$ m/s

This is a max velocity.

At time t, the velocity of a particle moving in a straight line is given by $v = \cos(t) + \sqrt{3}\sin(t) - 1$. For what value of t does it attain its max speed of 3?

$v = \cos(t) + \sqrt{3}\sin(t) - 1, \; t \geq 0.$ Max speed of 3 m/s.

$v = \cos t + \sqrt{3}\sin(t) - 1 = \sqrt{1+3}\left(\frac{1}{\sqrt{1+3}}\cos t + \frac{\sqrt{3}}{\sqrt{1+3}}\sin t\right) - 1$

$= 2\left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\sin t\right) - 1.$

let $\sin\varphi = \frac{1}{2}, \Rightarrow \varphi = \frac{\pi}{6} \quad \therefore \cos\varphi = \frac{\sqrt{3}}{2}.$

$\therefore v = 2(\sin\varphi\cos t + \cos\varphi\sin t) - 1 = 2\sin(\varphi + t) - 1$

$= 2\sin\left(t + \frac{\pi}{6}\right) - 1.$

$\therefore$ Max speed first reached when $\sin\left(t + \frac{\pi}{6}\right) = -1 \Rightarrow t + \frac{\pi}{6} = \frac{3\pi}{2}$

$\therefore t = \frac{4\pi}{3}$

$\therefore$ (D).

Equations for Motion with Constant Acceleration

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$x = \frac{1}{2}(u+v)t$$

Alternative Expressions for Acceleration

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

These can be used when solving different differential equations.

Example

If $a = v - 2$, and $v = 3$ when $x = 0$, find x when $v = 5$

$$a = v\frac{dv}{dx} = v - 2 \Rightarrow \frac{dv}{dx} = 1 - \frac{2}{v} = \frac{v-2}{v}$$

$$\therefore \frac{dx}{dv} = \frac{v}{v-2} \Rightarrow x = \int \frac{v}{v-2}\,dv = \int \frac{v-2+2}{v-2}\,dv$$

$$= \int\left(1 + \frac{2}{v-2}\right)dv$$

$$= v + 2\ln|v-2| + c$$

$$v=3,\ x=0 \quad \therefore 0 = 3 + \cancel{2\ln|1|} + c = 3 + c$$

$$\Rightarrow c = -3$$

$$\therefore x = v + 2\ln|v-2| - 3$$

$$\therefore x\big|_{v=5} = 5 + 2\ln|5-2| - 3 = 2 + 2\ln 3.$$

Velocity-Time Graphs

Velocity-time (v-t) graphs can be used to solve many kinematics problems. Note that:

- Area under $v - t$ graph is displacement
- Gradient of $v - t$ graph is acceleration

Vector Functions

A **vector valued function** is a function that maps to a set of vectors in $\mathbb{R}^n$. In SM, only the cases $\mathbb{R} \to \mathbb{R}^2$ and $\mathbb{R} \to \mathbb{R}^3$ are considered, making it a very simple topic.

Vector Limits

The limit of a vector function $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$ is defined as:

$$\boxed{\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} x(t)\vec{\imath} + \lim_{t\to a} y(t)\vec{\jmath} + \lim_{t\to a} z(t)\vec{k}}$$

Space Curves

Any continuous vector function traces out a space curve with the tip of its vector.

Examples:

- $\vec{r}(t) = (1+2t)\vec{\imath} + (3-t)\vec{\jmath} + (6+t)\vec{k}$ is a line passing through $(1,3,6)$ parallel to $(2,-1,1)$.
- $\vec{r}(t) = \cos(t)\,\vec{\imath} + \sin(t)\,\vec{\jmath} + t\vec{k}$ is a helix on the cylinder $x^2 + y^2 = 1$.

Conversion between Parametric and Cartesian

The aim here is to eliminate the parameter, to obtain a relation between x and y.

Example:
If $\vec{r}(t) = 3\sin(2t)\,\vec{\imath} + 3\cos(2t)\vec{\jmath}$, find the Cartesian form of the path of this vector function.

$$\begin{cases} x = 3\sin(2t) & \Rightarrow \frac{x}{3} = \sin(2t) \\ y = 3\cos(2t) & \Rightarrow \frac{y}{3} = \cos(2t) \end{cases}$$

$$\sin^2(2t) + \cos^2(2t) = \frac{x^2}{9} + \frac{y^2}{9} = 1 \Rightarrow x^2 + y^2 = 9$$

$x \in [-3, 3]$ $y \in [-3, 3]$.

Derivative

The derivative of a vector function $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$, denoted $\dot{\vec{r}}(t)$, is defined as:

$$\frac{d}{dt}(\vec{r}(t)) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t) = x'(t)\vec{\imath} + y'(t)\vec{\jmath} + z'(t)\vec{k}$$

Antiderivative

The antiderivative of a vector function $\vec{r}(t)$ is defined as:

$$\int \vec{r}(t)dt = \int x(t)dt\,\vec{\imath} + \int y(t)dt\,\vec{\jmath} + \int z(t)dt\,\vec{k} + \vec{C}$$

Where $\vec{C}$ is an arbitrary constant vector.

Fundamental Theorem of Vector Calculus

$$\int_a^b \vec{r}(t)dt = \vec{R}(b) - \vec{R}(a)$$

Example:

For the particle with position $\vec{r} = (t-3)\vec{\imath} - \sqrt{t}\vec{\jmath}$, find the time at which its distance from the origin is a minimum.

$\underset{\sim}{r}(t) = (t-3)\underset{\sim}{i} - \sqrt{t}\underset{\sim}{j}$

$\therefore |\underset{\sim}{r}(t)| = \sqrt{(t-3)^2 + t} = \sqrt{t^2 - 6t + 9 + t} = \sqrt{t^2 - 5t + 9}$

$\therefore$ Min of $t^2 - 5t + 9$ at : $t = \frac{-b}{2a} = \frac{5}{2(1)} = \frac{5}{2}$

$\therefore$ Min $|\underset{\sim}{r}(t)| = \sqrt{t^2 - 5t + 9}$ at $t = \frac{5}{2} = 2.5$ $\therefore$ B

Sketching Vector Functions

The path of a vector function can be sketched using the following steps:

- Find the Cartesian equation, and constraints on x/y.
- Find the initial position (and final position)
- Find initial direction

Example:

Sketch the path of the vector function $\vec{r} = 3\sin(2t)\,\vec{\imath} + 3\cos(2t)\,\vec{\jmath}$.

$\underset{\sim}{r}(t) = (3\sin(2t))\underset{\sim}{i} + (3\cos(2t))\underset{\sim}{j}, \quad t \geq 0.$

(a) $\dot{\underset{\sim}{r}}(t) = (6\cos(2t))\underset{\sim}{i} - (6\sin(2t))\underset{\sim}{j}$

(b) $\ddot{\underset{\sim}{r}}(t) = -12\sin(2t)\underset{\sim}{i} - 12\cos(2t)\underset{\sim}{j}$

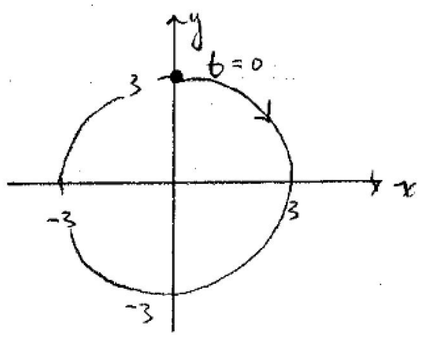
(c) $\begin{cases} x = 3\sin(2t) \Rightarrow \frac{x}{3} = \sin(2t) \\ y = 3\cos(2t) \Rightarrow \frac{y}{3} = \cos(2t) \end{cases}$

$\sin^2(2t) + \cos^2(2t) = \frac{x^2}{9} + \frac{y^2}{9} = 1 \Rightarrow x^2 + y^2 = 9$

$x \in [-3, 3] \quad y \in [-3, 3].$

(d)

$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 3\underset{\sim}{j}, \quad \dot{\underset{\sim}{r}}(0) = 6\underset{\sim}{i} + 0\underset{\sim}{j}$



Applications in Kinematics

In kinematics, the position vector of a particle at time t can be denoted $\vec{r}(t)$. The velocity vector is thus $d\vec{r}/dt$, and the acceleration vector $d^2\vec{r}/dt^2$. To find the magnitude of velocity, the magnitude of $\dot{\vec{r}}(t)$ is taken, i.e. $|\dot{\vec{r}}(t)|$. The magnitude of acceleration is likewise $|\ddot{\vec{r}}(t)|$.

Example:

The acceleration vector of a particle at time $t \geq 0$ is given by $\vec{a} = 2\sin(t)\,\vec{\imath}$. The velocity of the particle when $t = \pi$ is $2\vec{\imath} + 2\vec{\jmath}$. Find the initial velocity of the particle.

$\underset{\sim}{a}(t) = 2\sin(t)\underset{\sim}{i}$, $\underset{\sim}{v}(\pi) = 2\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\, dt = -2\cos(t)\underset{\sim}{i} + \underset{\sim}{c}$

$\underset{\sim}{v}(\pi) = -2\cos(\pi)\underset{\sim}{i} + \underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{c} = 2\underset{\sim}{i} + 2\underset{\sim}{j} \Rightarrow \underset{\sim}{c} = 2\underset{\sim}{j}$.

$\therefore \underset{\sim}{v}(t) = -2\cos(t)\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{v}(0) = -2\underset{\sim}{i} + 2\underset{\sim}{j} \quad \therefore C$

Dynamics

Dynamics is the study of a mathematical model for motion. This topic is purely mathematical physics, and is quite simple.

Units

Length	Metre (m)
Time	Second (s)
Mass	Kilogram (kg)

Acceleration due to Gravity

$$g = 9.8\text{m/s}^2 \text{ near the surface of the Earth}$$

Momentum

A physical quantity interpreted as "quantity of movement", denoted p. Measured in kg m s^{-1}, a **vector quantity**.

$$\vec{p} = m\vec{v}$$

Force

A physical quantity that causes movement. Measured in Newtons (N), 1 N = 1 kg m s^{-2}.

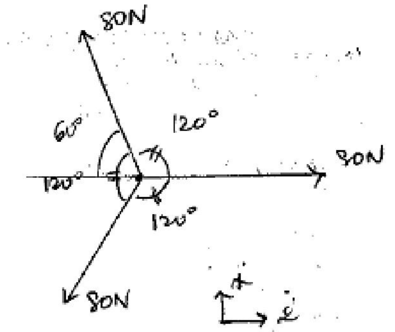
$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Resolution of Forces

Can be done in two ways, depending on which one is easier.

- Geometrically, $F_x = F\cos(\theta)\,, F_y = F\,\sin(\theta)$, where θ is the angle between the vector and the positive x-axis.
- Using scalar/vector resolutes, the scalar resolute of F in the $\vec{\imath}$ direction is $\vec{F}.\vec{\imath}$, and $\vec{F}.\vec{\jmath}$ in the $\vec{\jmath}$ direction.

Example:
An object experiences 3 forces of 80N, with 120° between adjacent pairs of forces. Draw the force diagram, resolve forces, and find the magnitude of the resultant force.



$$\sum \underset{\sim}{F} = (80 - 80\cos 60° - 80\cos 60°)\underset{\sim}{i} + (80\sin 60° - 80\sin 60°)\underset{\sim}{j}$$
$$= (80 - 160\cos 60°)\underset{\sim}{i} + 0\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$$

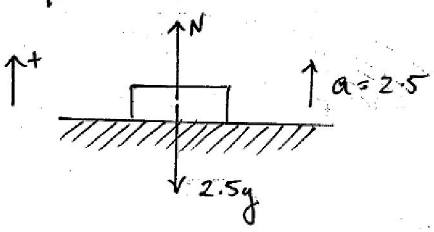
Newton's Laws

- **Newton's First Law of Motion** – a particle remains stationary or in uniform rectilinear motion unless acted on by a net external force.
- **Newton's Second Law of Motion** – a particle acted on by a net force will move in such a way that the rate of change of momentum with respect to time is proportional to the force.
- **Newton's Third Law of Motion** – for every action force, there is an equal and opposite reaction force exerted. **Important: these forces are NOT on the same object. E.g. if A exerts a force F on B, then B exerts a force -F on A.**

Normal Reaction Force

The normal reaction force, $\overrightarrow{F_n}$, is the reaction force exerted by a surface on a particle. By Newton III, it is equal and opposite to the force exerted on the surface by the particle.
Example

A box is on the floor of a lift accelerating at 2.5ms^{-2} upwards. The mass of the box is 2.5kg. Find the reaction force of the floor on the box.



$$\sum F = N - 2.5g = (2.5)(2.5)$$
$$\therefore N = (2.5)^2 + 2.5g = 30.75\ (N) \text{ up.}$$

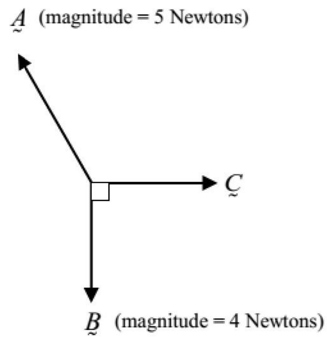
Finding Resultant Forces

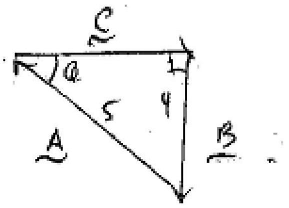
The resultant force can be calculated geometrically by finding components or using triangle geometry (in the case of equilibrium system), or with vector notation.

$$\overrightarrow{F_{net}} = \sum_{i=1}^{n} \overrightarrow{F_i}$$

Example:

For the equilibrium system shown, find the magnitude of C.





$\therefore |\underset{\sim}{C}| = \sqrt{25-16} = \sqrt{9} = 3$

Sliding Friction

A mathematical model for friction is given by the equation

$$F_r \leq \mu F_n$$

Where F_n is the normal reaction force, F_r is the force due to friction, and μ is the **coefficient of kinetic friction**.

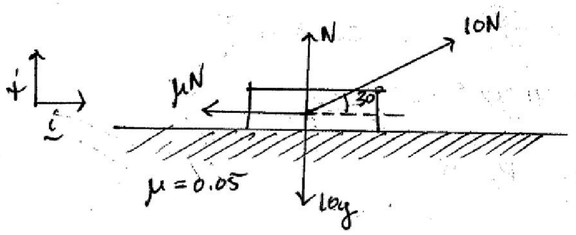
When $F_r = \mu F_n$, the system is either moving against friction or **on the verge of doing so.**

If a surface is said to be **smooth**, then μ=0 ⇒ $F_r = 0$.

Friction always acts in the direction which **opposes the direction of motion,** i.e. **opposite the direction of $\Sigma\vec{F}$.**

Example

a block of mass 10kg is being pulled as follows. Find its acceleration.



$\underset{\sim}{i}$: $10\cos 30° - \mu N = 10a$

$\underset{\sim}{j}$: $10\sin 30° + N - 10g = 0$

$\therefore N = 10g - 10\sin 30°$

$= 10g - 5$

$\therefore \frac{10\sqrt{3}}{2} - \mu(10g - 5) = 10a$

$\therefore 5\sqrt{3} - 0.05(10g - 5) = 10a$

$\therefore a = \frac{5\sqrt{3} - 0.05(10g-5)}{10} \approx 0.40\,\text{ms}^{-2}$

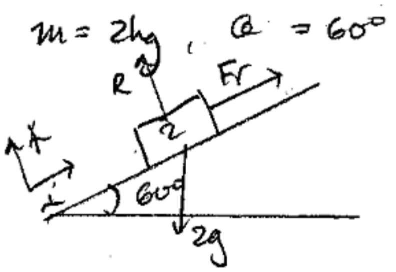
Forces on an Inclined Plane

Some useful equations for this common case include:

- **Force on particle down slope:** $F_{parr} = F_g \sin(\theta)$
- **Force on particle perpendicular to slope:** $F_{perp} = F_g \cos(\theta)$

- **Normal force on particle:** $F_n = F_{perp} = F_g \cos(\theta)$ (assuming zero acceleration for the entire system)

Example:
A particle of mass 2kg rests on a rough plane inclined at 60° to the horizontal. Find the frictional force preventing the mass from slipping.



$\swarrow$: $F_r - 2g \sin 60° = 0$

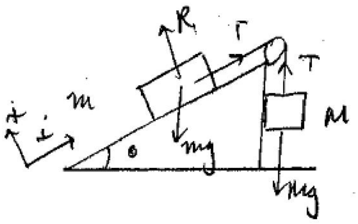
$F_r = 2g \sin 60° = 2g \times \frac{\sqrt{3}}{2} = \sqrt{3}g$

∴ D.

Connected Particles
Unlike in Physics, it is not a good idea to consider first the **net force on the system**, and then to consider the forces acting on each object. Instead, consider each particle alone, and write an equation of motion for it, setting up a system of equations. From this system, an unknown force such as tension can be eliminated.

- **Tension** in a connection (e.g. string) is assumed to be **uniform** and always acts towards the **centre** of the string.

Example:
For the shown situation, show that $M = m \sin(\theta)$.



M: $T - Mg = 0 \Rightarrow T = Mg$ (2)

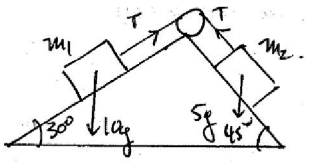
m: i: $T - mg\sin\theta = 0$. (1)

j: $R - mg\cos\theta = 0$.

(1) − (2): $\cancel{T} - mg\sin\theta - \cancel{T} = -Mg$

$Mg = mg\sin\theta$

$\therefore M = m\sin\theta$ $\therefore$ B



Find the acceleration of the system if the inclines are smooth, and the tension in the string.

Parallel to tension in string: m_1: $10g\sin 30^\circ - T = 10a$ —— (1)

m_2: $5g\sin 45^\circ - T = -5a$ —— (2)

(1) − (2): $\frac{10g}{2} - \cancel{T} - \frac{5g}{\sqrt{2}} + \cancel{T} = 10a + 5a$

$5g - \frac{5\sqrt{2}g}{2} = 15a$

$a = \frac{1}{15}\left(5g - \frac{5\sqrt{2}}{2}g\right) \approx 0.9568 \text{ ms}^{-2}$ (4dp)

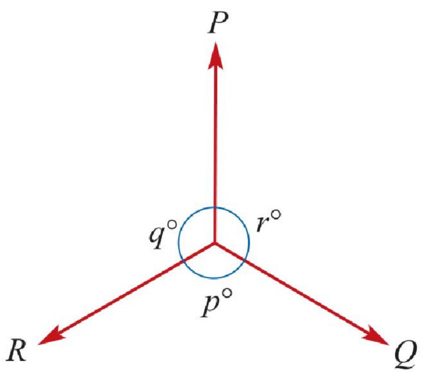
$T = 10g\sin 30^\circ - 10a = 5g - \frac{10}{15}\left(5g - \frac{5\sqrt{2}}{2}g\right) \approx 39.4322$ N (4dp).

Equilibrium

Equilibrium is the study of cases where the resultant force is zero, i.e. when there is no movement.

Lami's Theorem

Lami's Theorem states, for an equilibrium system with 3 forces P, Q, R:

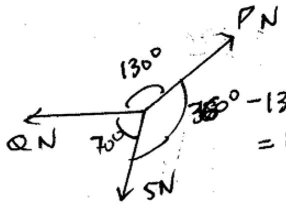


$$\frac{P}{\sin(p)} = \frac{Q}{\sin(q)} = \frac{R}{\sin(r)}$$

Where p, q, r are the angles opposite P, Q, R respectively.

Example

Use Lami's Theorem to find P, Q



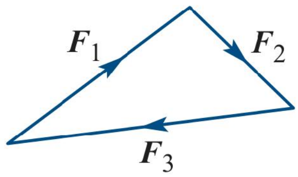
$360° - 130° - 70° = 160°$

$$\therefore \frac{P}{\sin 70°} = \frac{Q}{\sin 160°} = \frac{5}{\sin 130°}$$

$$\therefore P = \frac{5\sin 70°}{\sin 130°} \approx 6.133 \text{ (N)}$$

$$Q = \frac{5\sin 160°}{\sin 130°} \approx 2.232 \text{ (N)}$$

Triangles of Forces

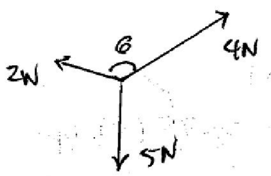


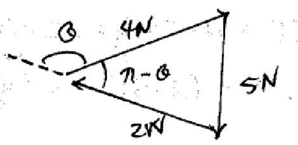
For the case of an equilibrium system of 3 forces, the 3 forces can be joined to form a **triangle of forces**. (Think about a vector analysis of these forces, and the addition of three vectors to form a triangle ⇒ their sum is zero)

Analysis of these cases using triangles of vectors typically involve geometric theorems such as the sine and cosine rules.

Example

Find θ, if the system is in equilibrium.





(NOT TO SCALE)

$\therefore\ 5^2 = 4^2 + 2^2 - 2(4)(2)\cos(\pi-\theta)$

$25 = 16 + 4 + 16\cos\theta$

$\therefore\ 25 = 20 + 16\cos\theta$

$\cos\theta = \frac{25-20}{16} = \frac{5}{16}. \quad \therefore\ \cos\theta = \frac{5}{16}$

$\therefore\ \theta = \arccos\left(\frac{5}{16}\right) \approx 71.79^\circ$